UNITARY CHANNEL DISCRIMINATION BEYOND GROUP STRUCTURES:

ADVANTAGES OF SEQUENTIAL AND INDEFINITE CAUSAL ORDER STRATEGIES

JESSICA BAVARESCO, MIO MURAO, MARCO TÚLIO QUINTINO

J. Math. Phys. 63, 042203 (2022), arXiv:2105.13369 [quant-ph]
Phys. Rev. Lett 127, 200504 (2021), arXiv:2011.08300 [quant-ph]

UNIVERSITÉ DE GENÈVE

Fonds national suisse

THE TASK:
MINIMUM-ERROR CHANNEL DISCRIMINATION

CHANNEL DISCRIMINATION

INPUT:

CHANNEL DISCRIMINATION

INPUT:

CANDIDATES: $\widetilde{C}_1$, $\widetilde{C}_2$, $\widetilde{C}_3$

CHANNEL DISCRIMINATION

INPUT: ?

CANDIDATES: $\widetilde{C}_1, \widetilde{C}_2, \widetilde{C}_3$

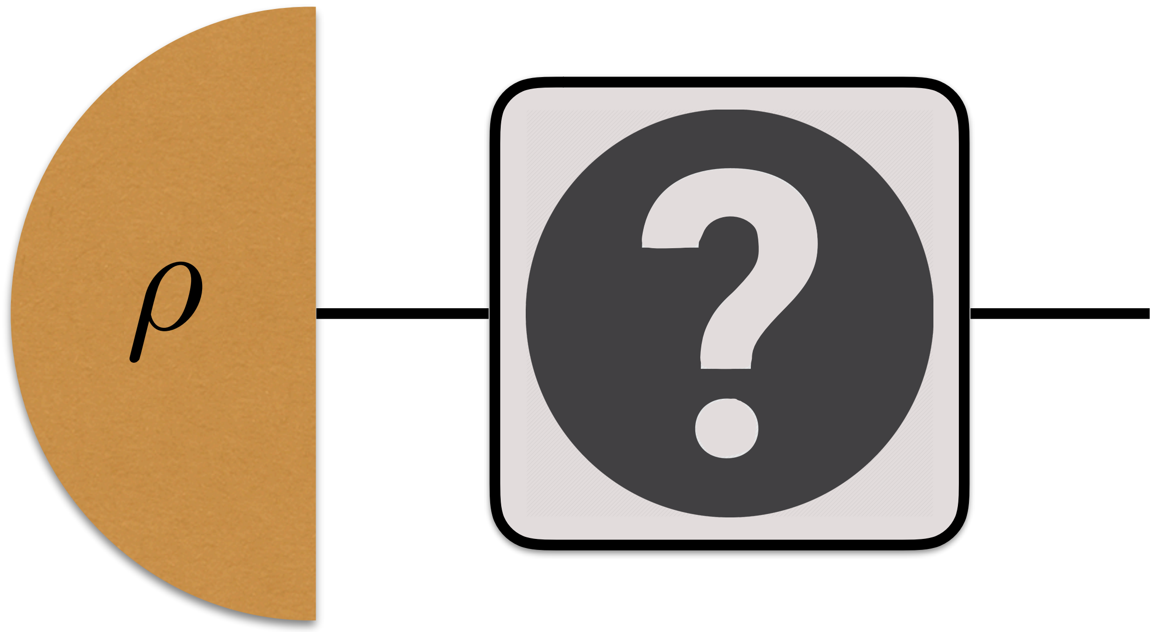
PROMISSE: $? = \widetilde{C}_1$, $? = \widetilde{C}_2$, $? = \widetilde{C}_3$

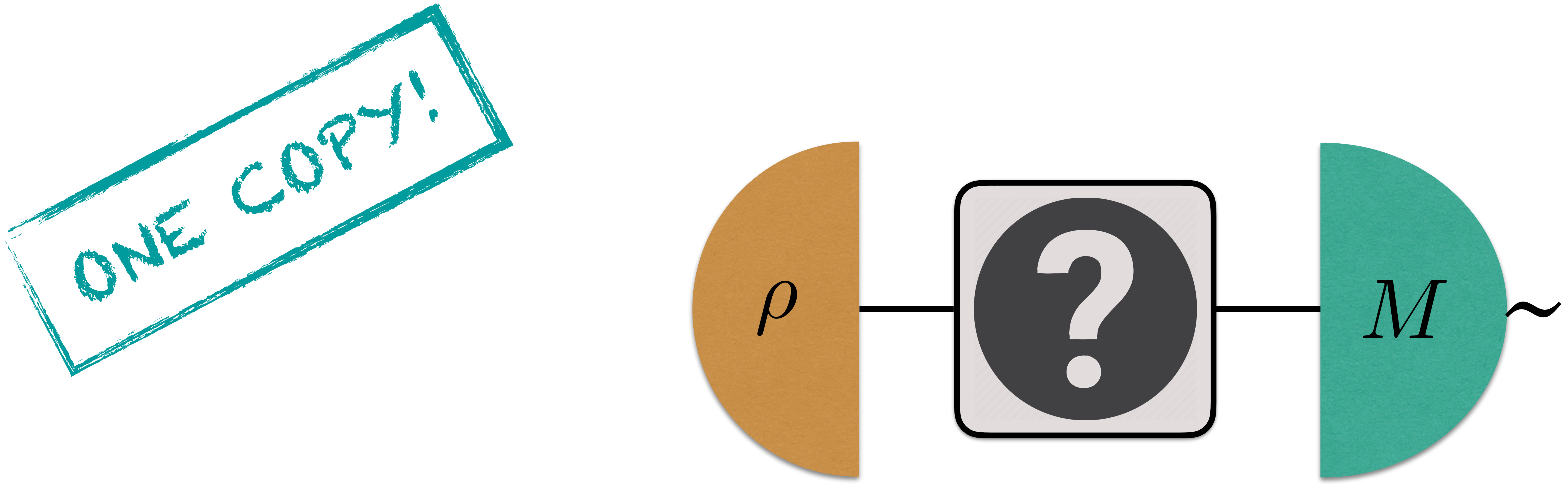
p_1 p_2 p_3

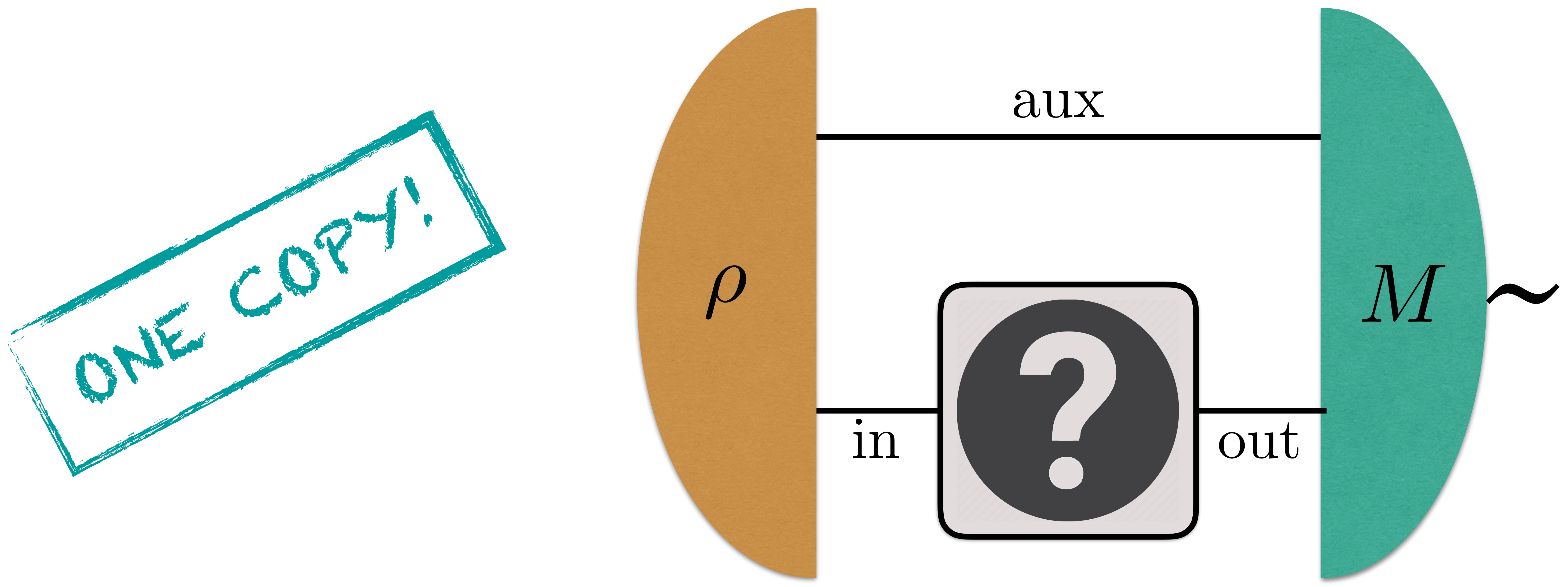
STRATEGY

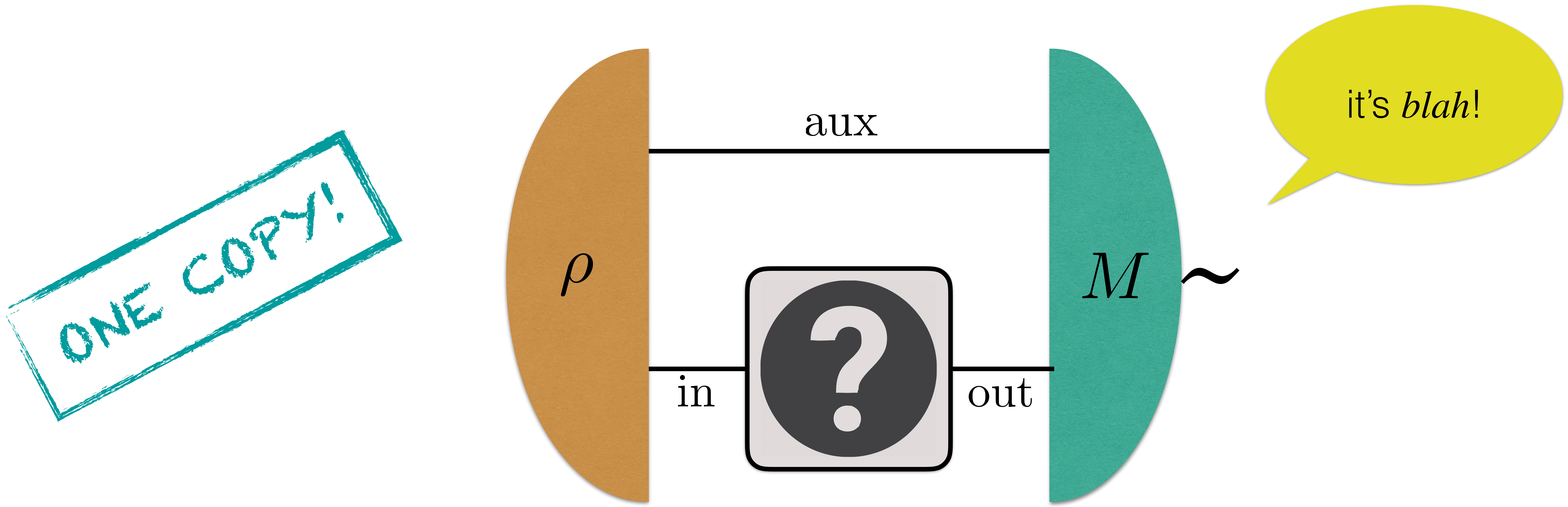
ONE COPY!

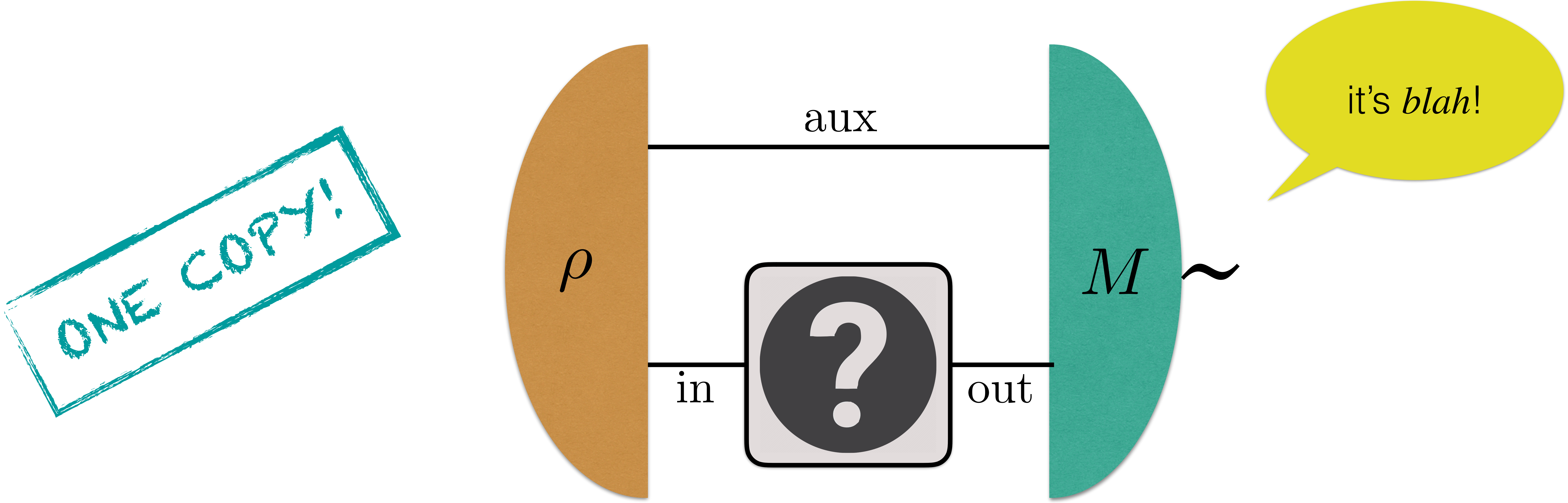
STRATEGY



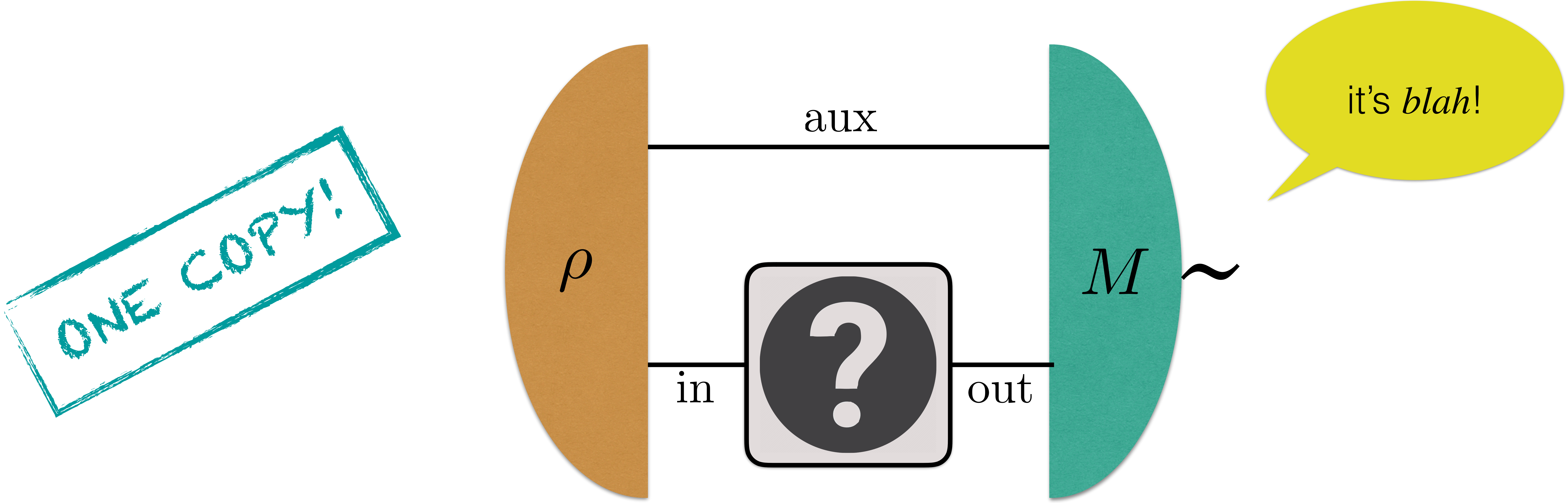

ONE COPY!
ρ
M


ONE COPY!
aux
ρ
M
~
in
out

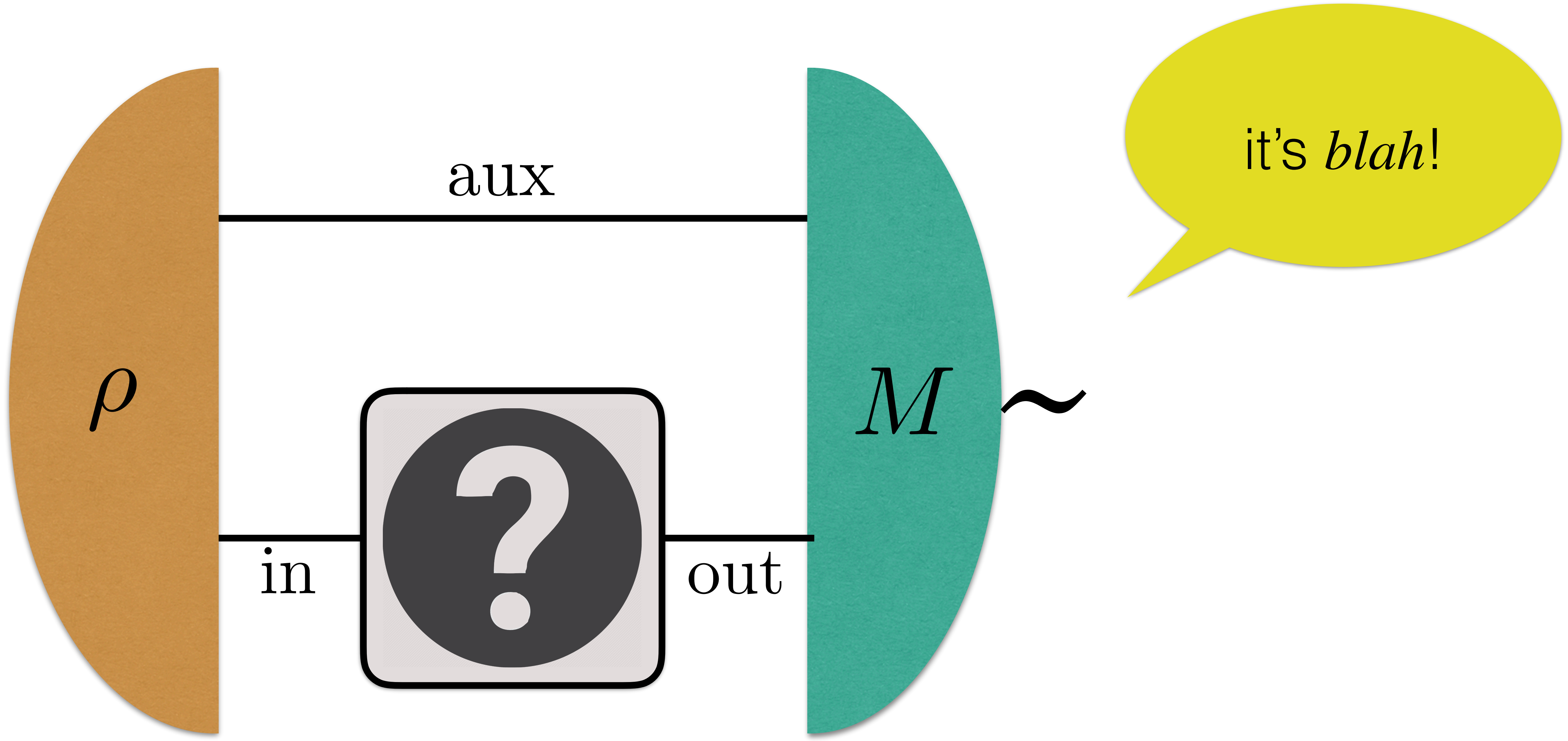

ONE COPY!
aux
ρ
in
out
M
~
it's *blah*!



$$p_{\text{succ}} = p_1\, p(1|\widetilde{C}_1, \rho, M) + p_2\, p(2|\widetilde{C}_2, \rho, M) + p_3\, p(3|\widetilde{C}_3, \rho, M)$$



$$
\begin{aligned}
p_{\mathrm{succ}} &= p_1\, p(1|\widetilde{C}_1, \rho, M) + p_2\, p(2|\widetilde{C}_2, \rho, M) + p_3\, p(3|\widetilde{C}_3, \rho, M) \\
&= \sum_{i=1}^{N} p_i \,\mathrm{Tr}[(\widetilde{C}_i \otimes \widetilde{\mathbb{I}})(\rho)\, M_i]
\end{aligned}
$$

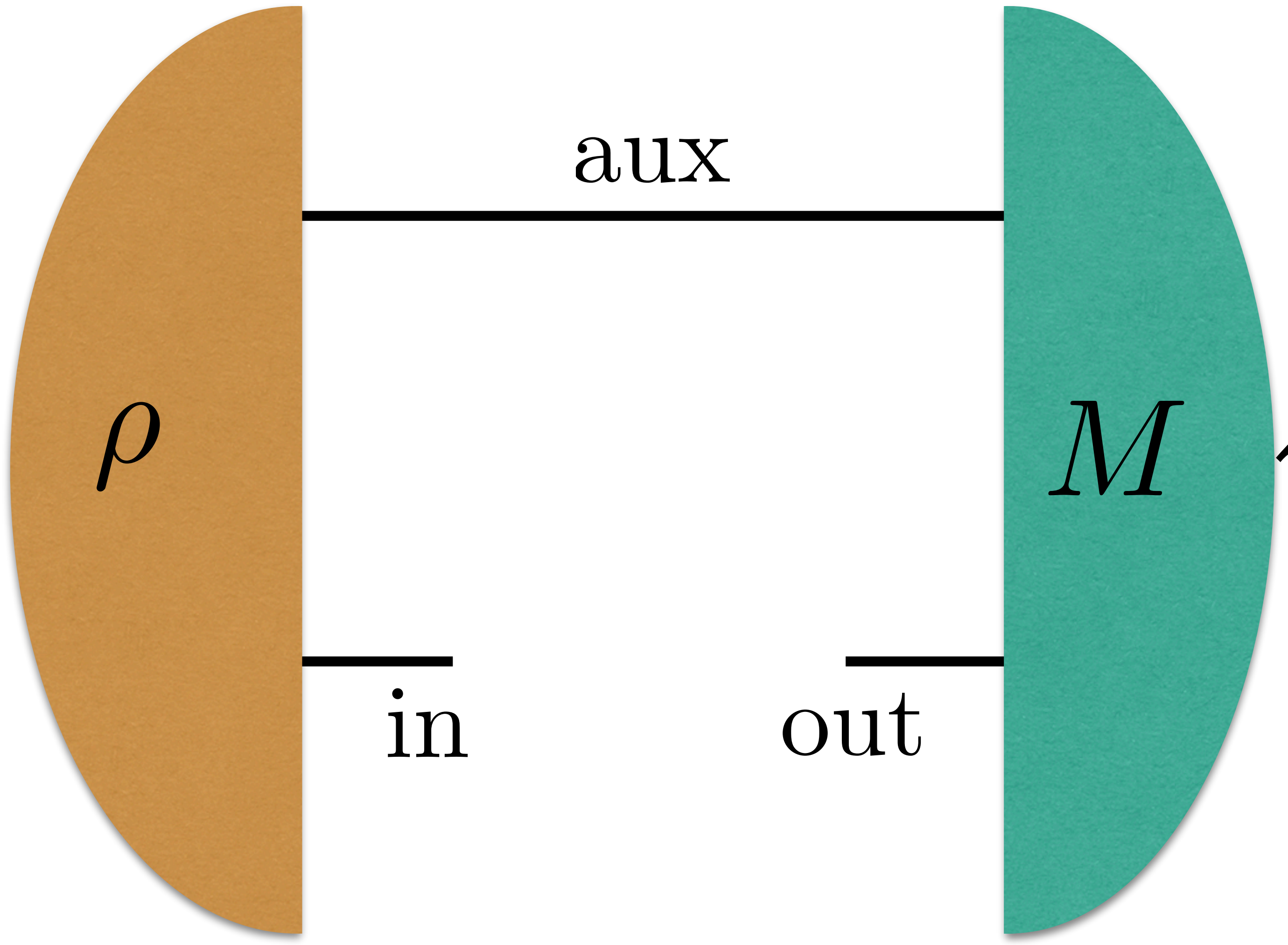


$$p_{\text{succ}} = p_1\, p(1|\widetilde{C}_1, \rho, M) + p_2\, p(2|\widetilde{C}_2, \rho, M) + p_3\, p(3|\widetilde{C}_3, \rho, M)$$

$$= \sum_{i=1}^{N} p_i \operatorname{Tr}[(\widetilde{C}_i \otimes \widetilde{\mathbb{I}})(\rho)\, M_i]$$

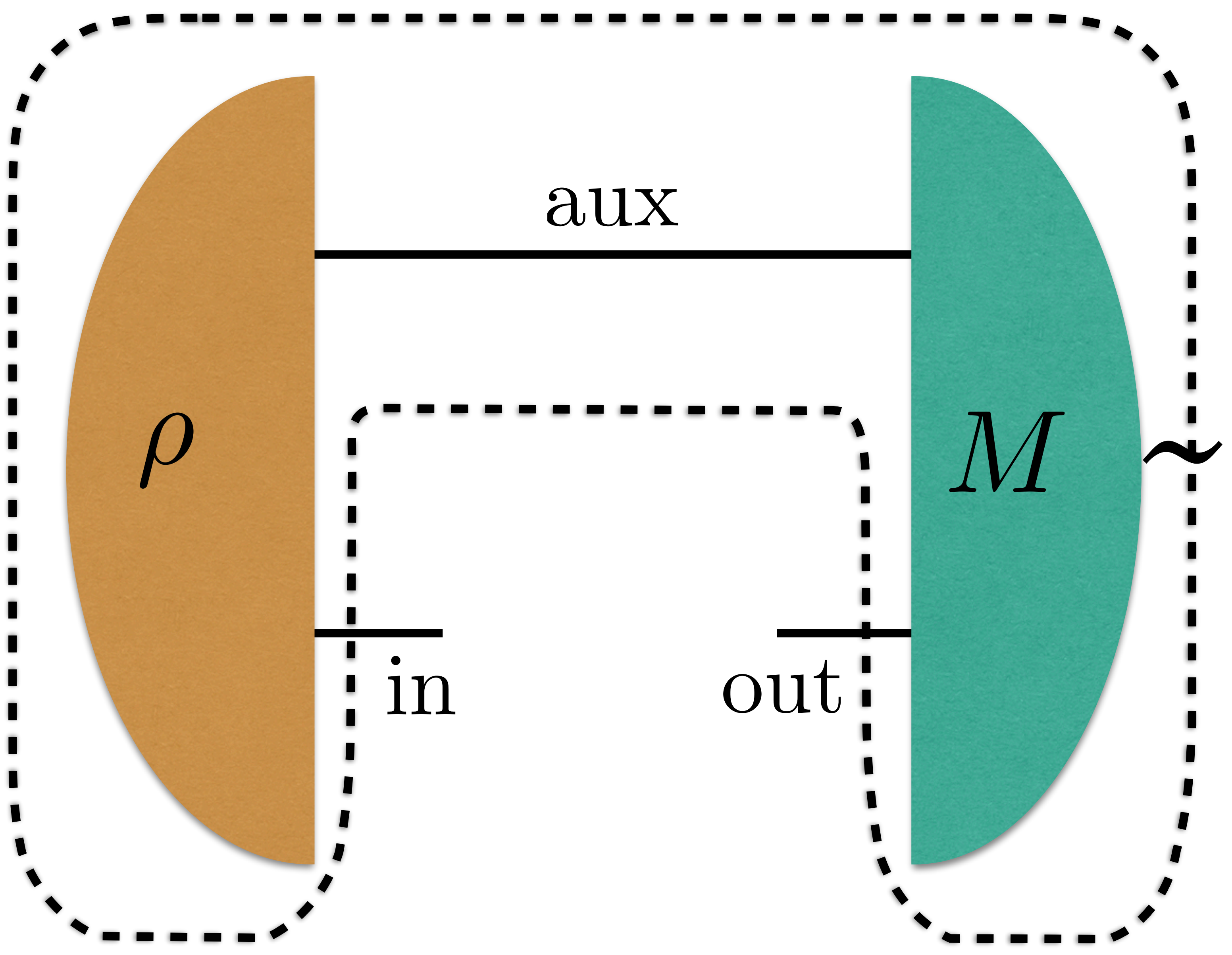
$$P := \underset{\rho, \{M_i\}}{\max} \sum_{i=1}^{N} p_i \operatorname{Tr}[(\widetilde{C}_i \otimes \widetilde{\mathbb{I}})(\rho)\, M_i]$$

HIGHER-ORDER OPERATIONS: TESTERS

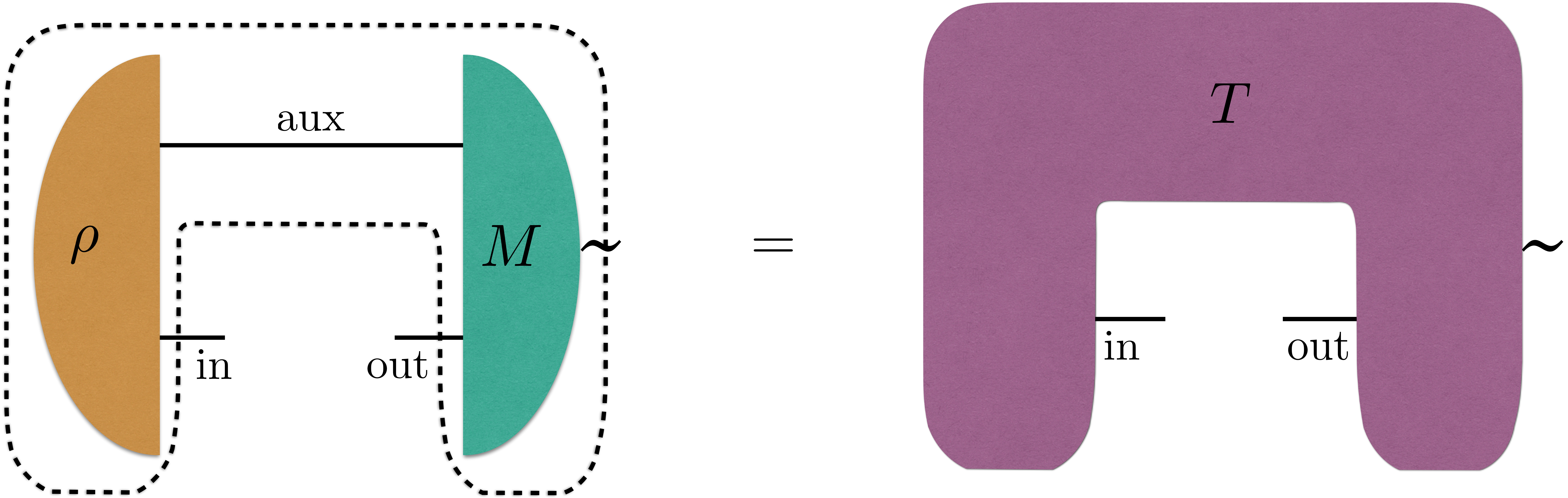


$\sim$

HIGHER-ORDER OPERATIONS: TESTERS

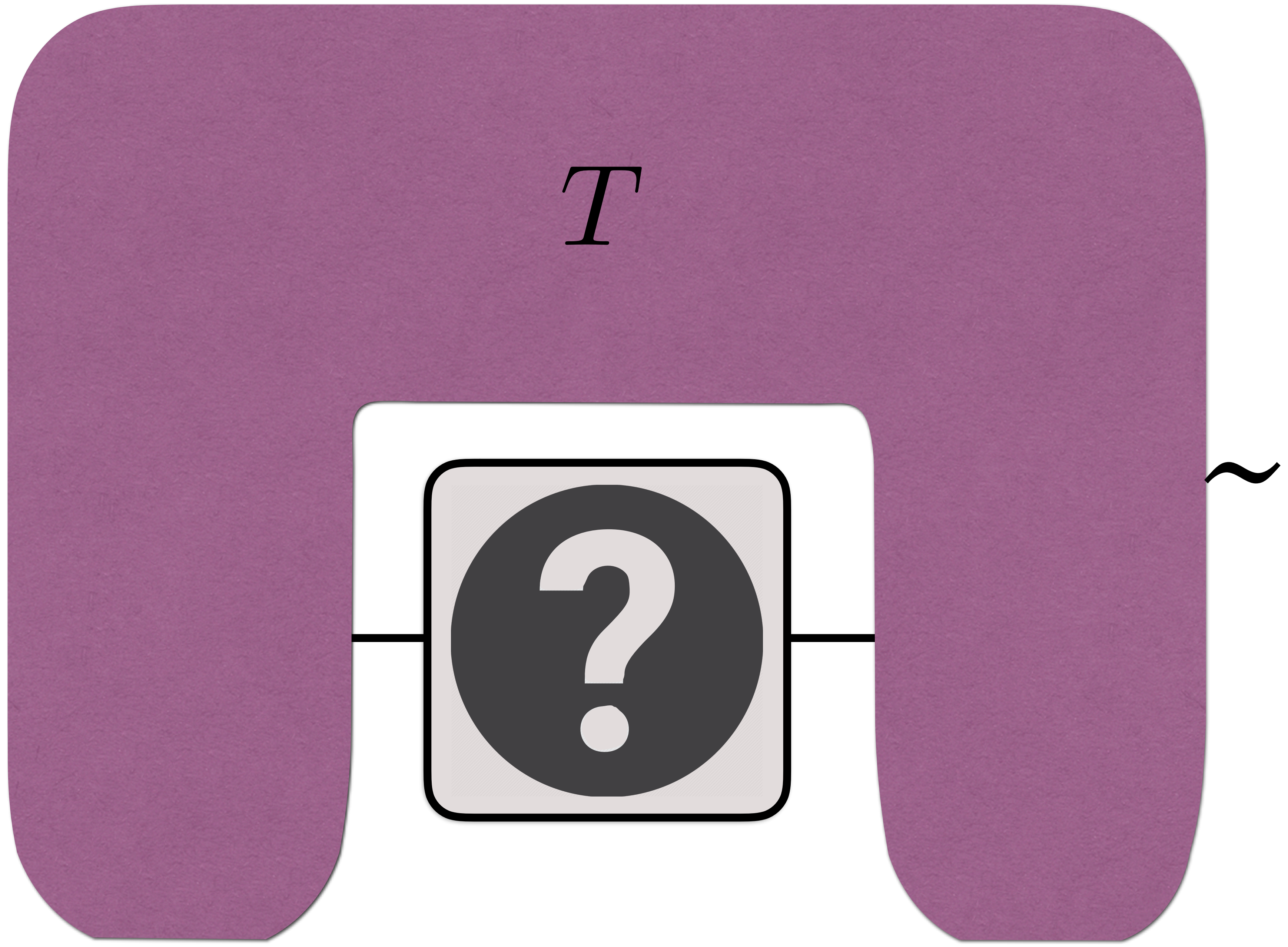


HIGHER-ORDER OPERATIONS: TESTERS



[1] G. Chiribella, G. M. D'Ariano, and P. Perinotti, PRL 101, 060401 (2008), arXiv:0712.1325 [quant-ph]
[2] G. Chiribella, G. M. D'Ariano, and P. Perinotti, PRL 101, 180501 (2008), arXiv:0803.3237 [quant-ph]
[3] M. Ziman, PRA 77, 062112 (2008), arXiv:0802.3862 [quant-ph]

HIGHER-ORDER OPERATIONS: TESTERS



$$p(i|C, T_i) = \mathrm{Tr}(C\,T_i)$$

[1] G. Chiribella, G. M. D'Ariano, and P. Perinotti, PRL 101, 060401 (2008), arXiv:0712.1325 [quant-ph]

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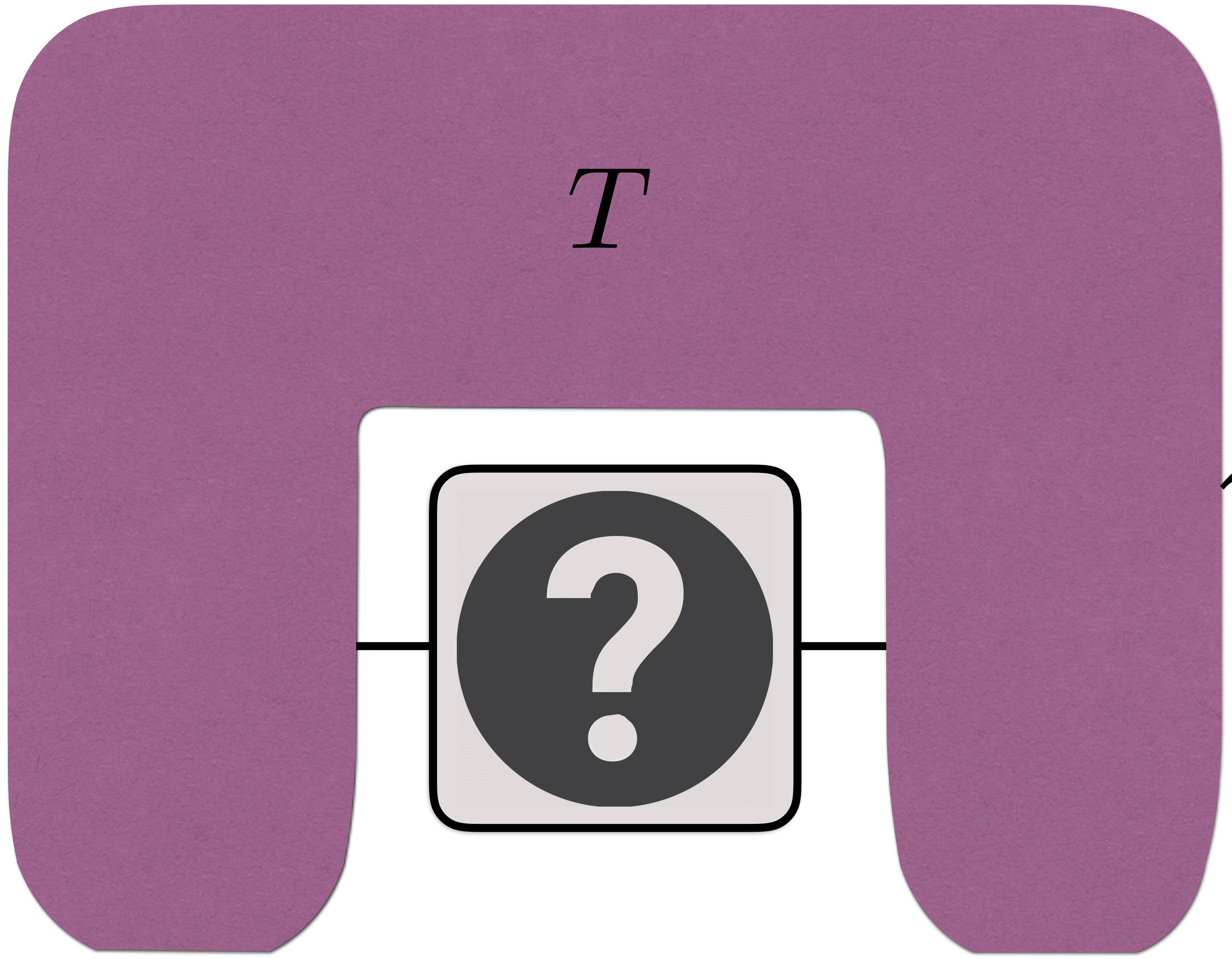
HIGHER-ORDER OPERATIONS: TESTERS



$\sim$

$$p(i|C, T_i) = \mathrm{Tr}(C\, T_i)$$

$$T = \{T_i\} \quad : T_i \in L(H^{\text{in, out}})$$

$$T_i \geq 0$$

$$\sum_i T_i = \sigma^{\text{in}} \otimes \mathbb{I}^{\text{out}}$$

[1] G. Chiribella, G. M. D'Ariano, and P. Perinotti, PRL 101, 060401 (2008), arXiv:0712.1325 [quant-ph]

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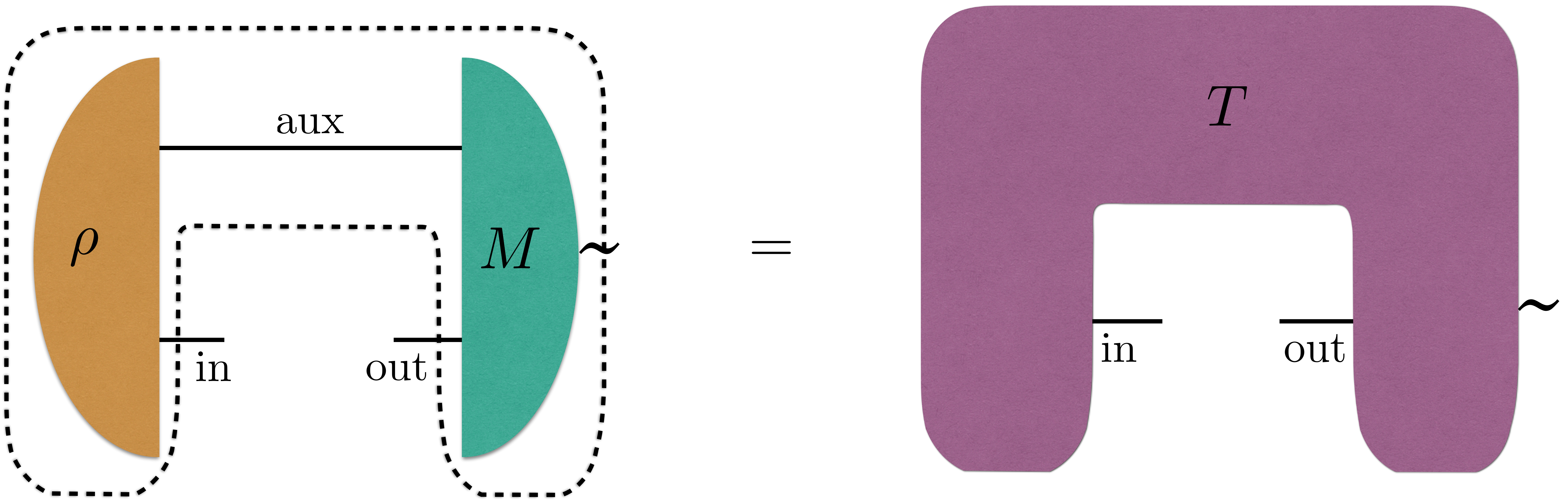
HIGHER-ORDER OPERATIONS: TESTERS

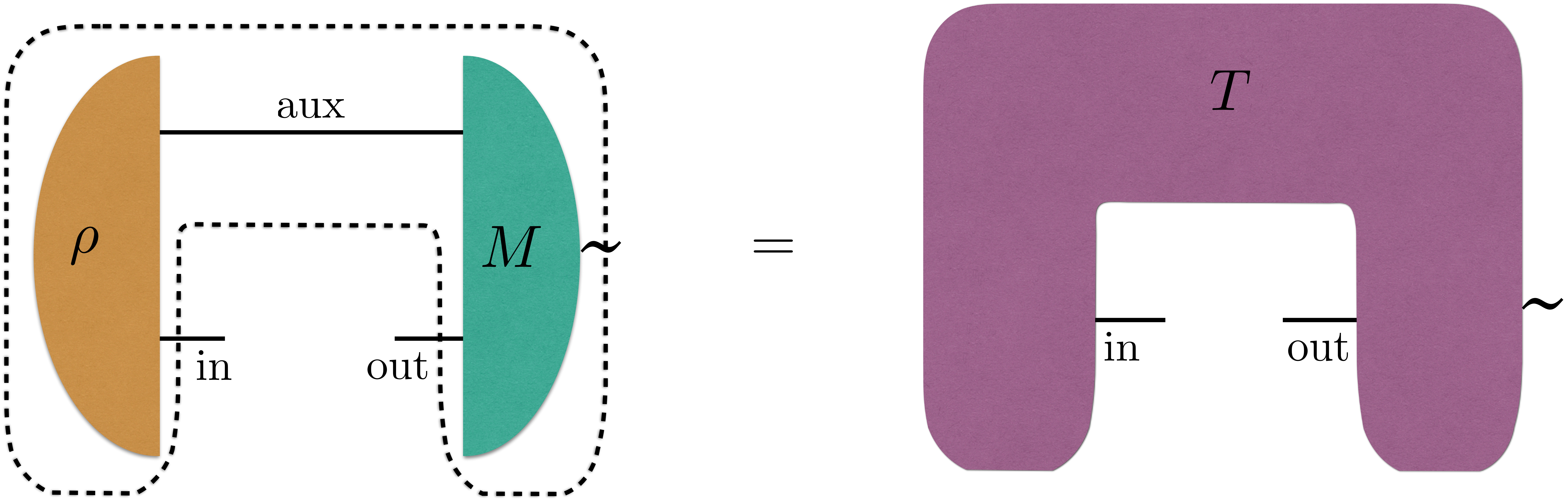


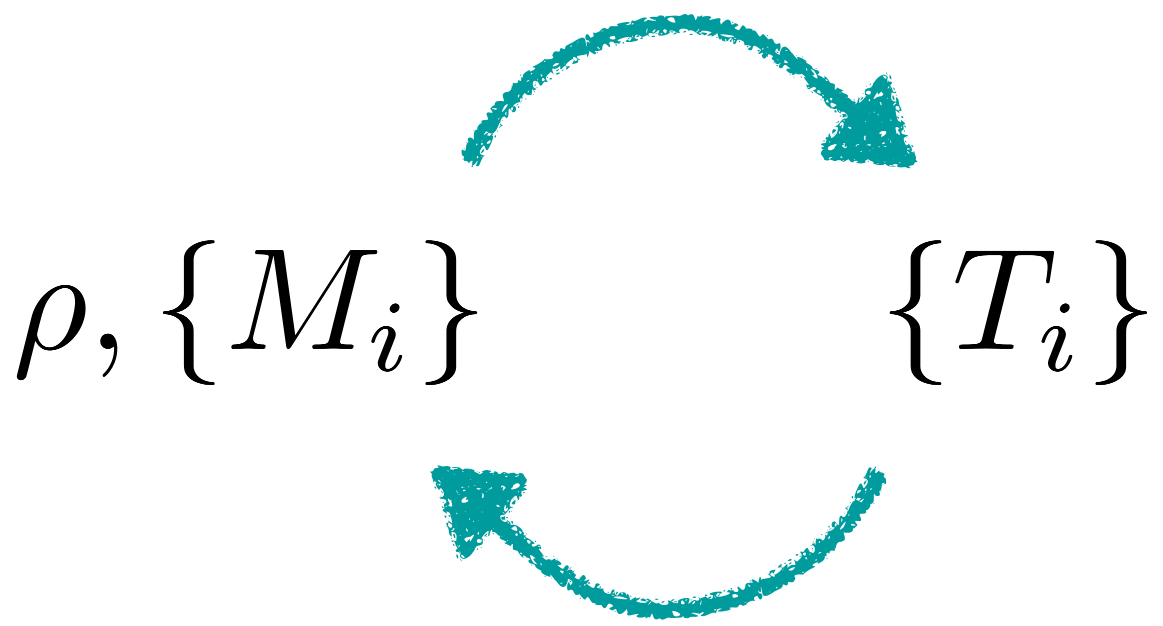
$$T_i = \mathrm{Tr}_{\mathrm{aux}}\left[(\rho^{\mathrm{in,aux}} \otimes \mathbb{I}^{\mathrm{out}})(\mathbb{I}^{\mathrm{in}} \otimes (M_i^{\mathrm{aux,out}})^{T_{\mathrm{out}}})\right]$$

$$\Longleftrightarrow$$

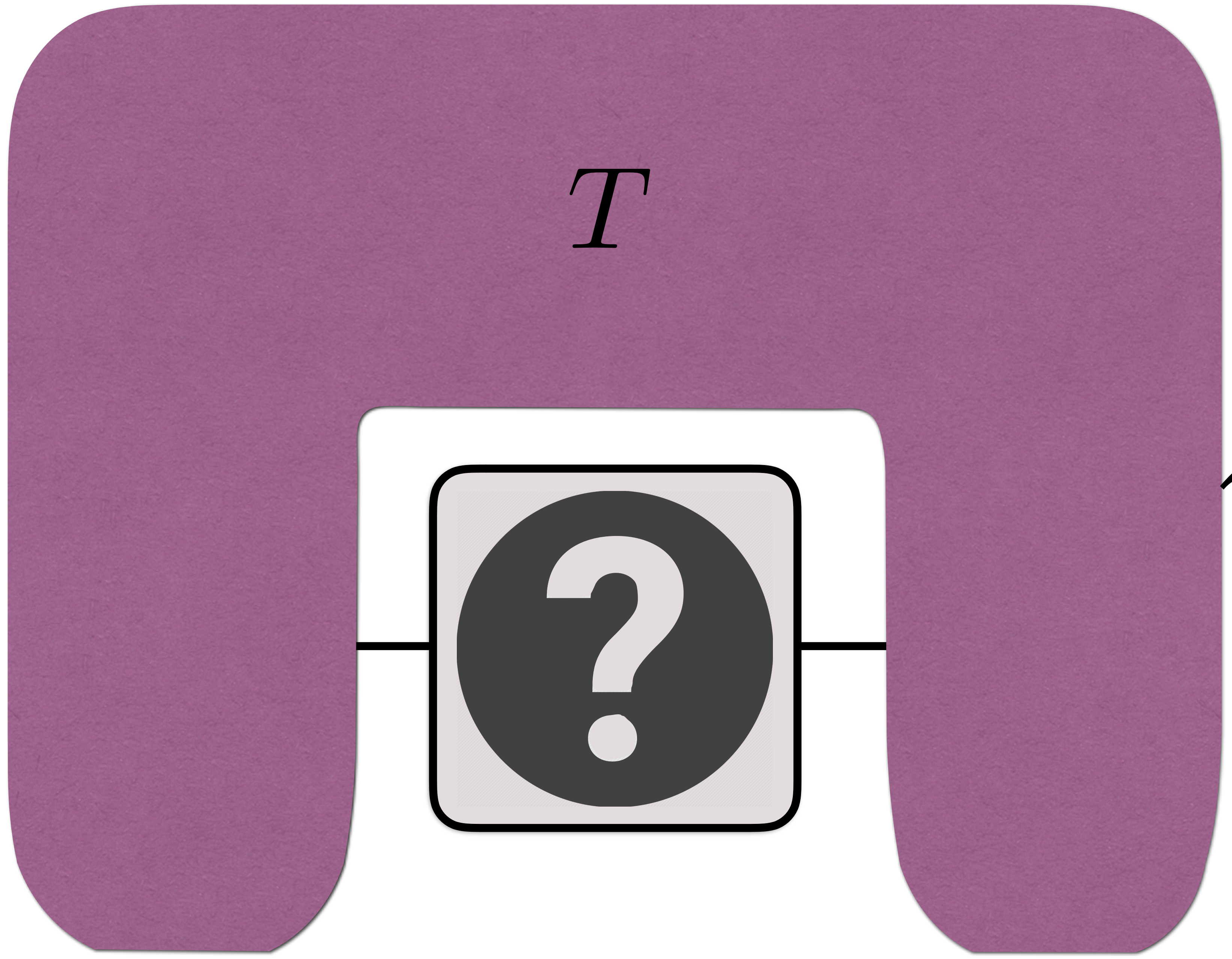
$$T_i \geq 0, \quad \sum_i T_i = \sigma^{\mathrm{in}} \otimes \mathbb{I}^{\mathrm{out}}$$

HIGHER-ORDER OPERATIONS: TESTERS





HIGHER-ORDER OPERATIONS: TESTERS



$\sim$

$$T = \{T_i\} \quad : T_i \in L(H^{\text{in, out}})$$

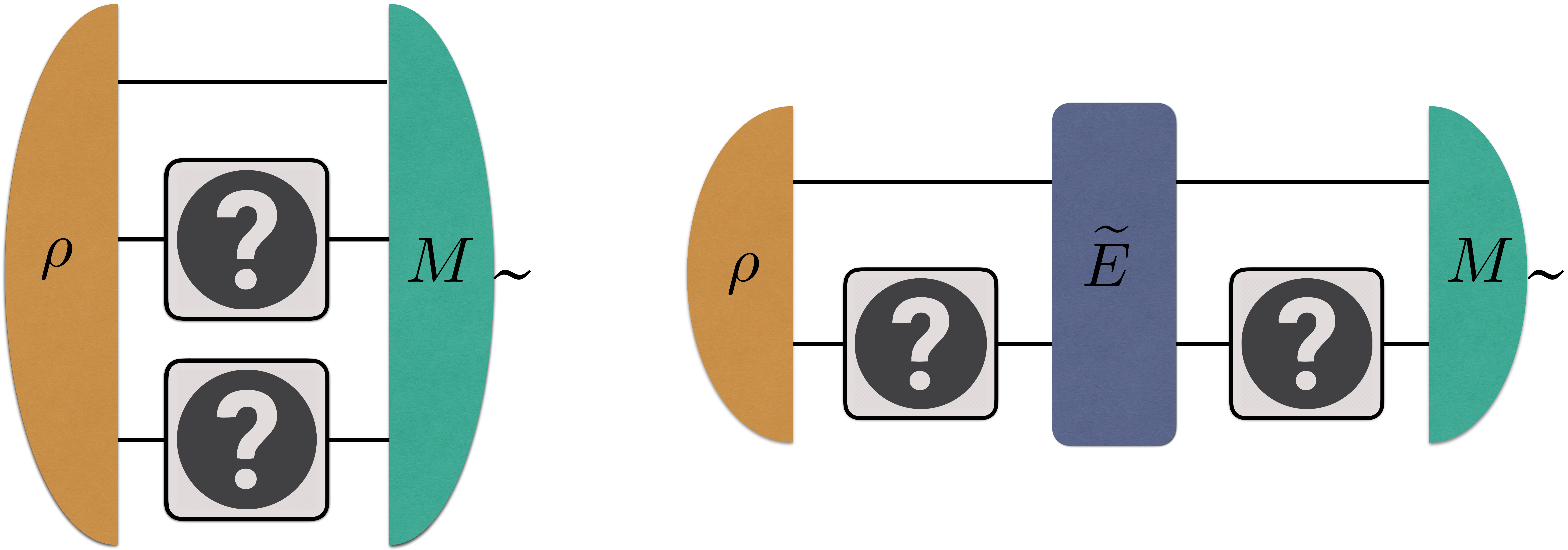
$$T_i \geq 0$$

$$\sum_i T_i = \sigma^{\text{in}} \otimes \mathbb{I}^{\text{out}}$$

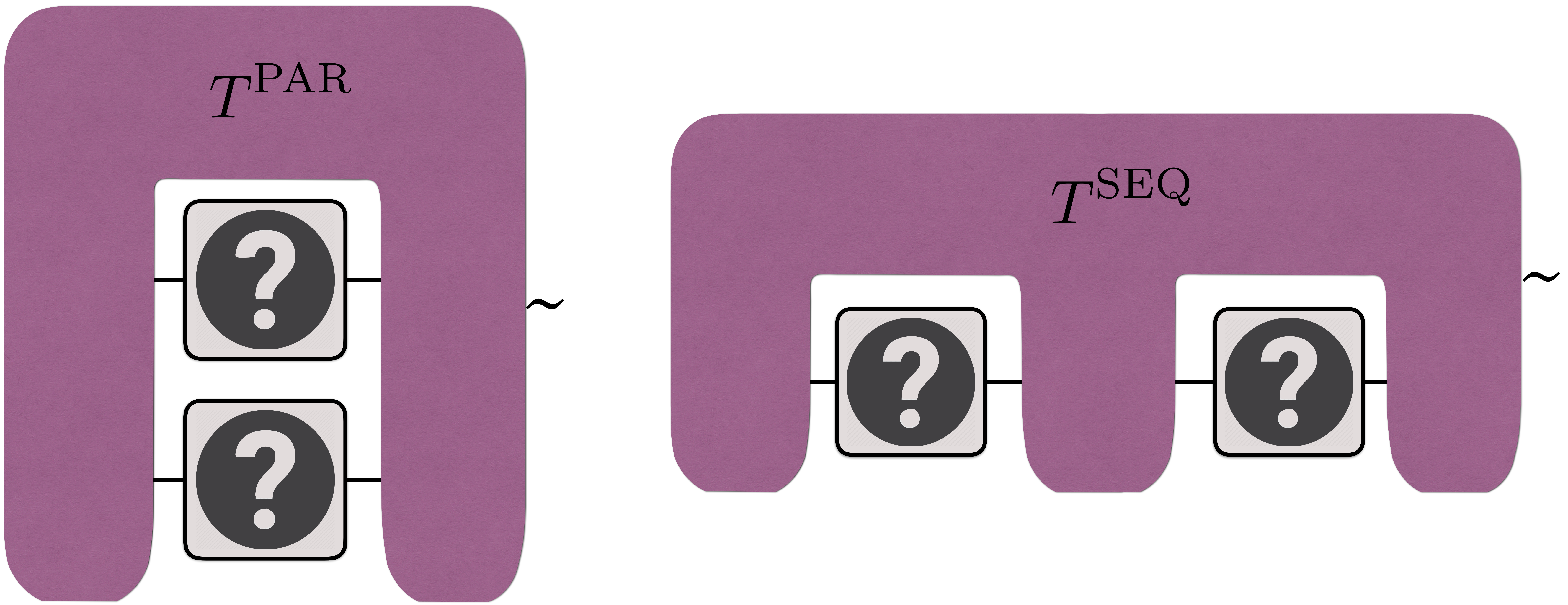
$$P = \max_{\{T_i\}} \sum_i p_i \, \mathrm{Tr}(C_i \, T_i)$$

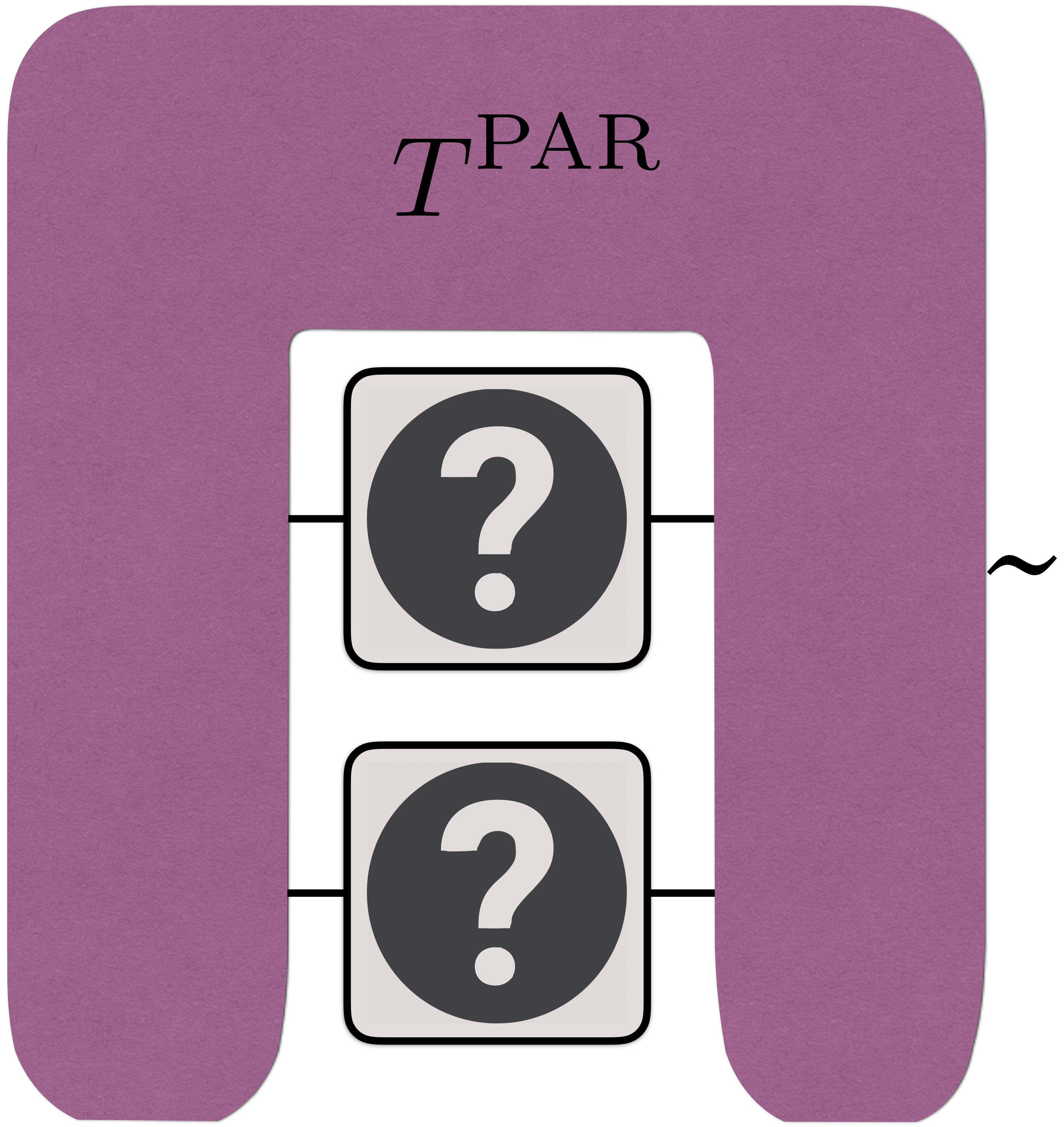
SDP

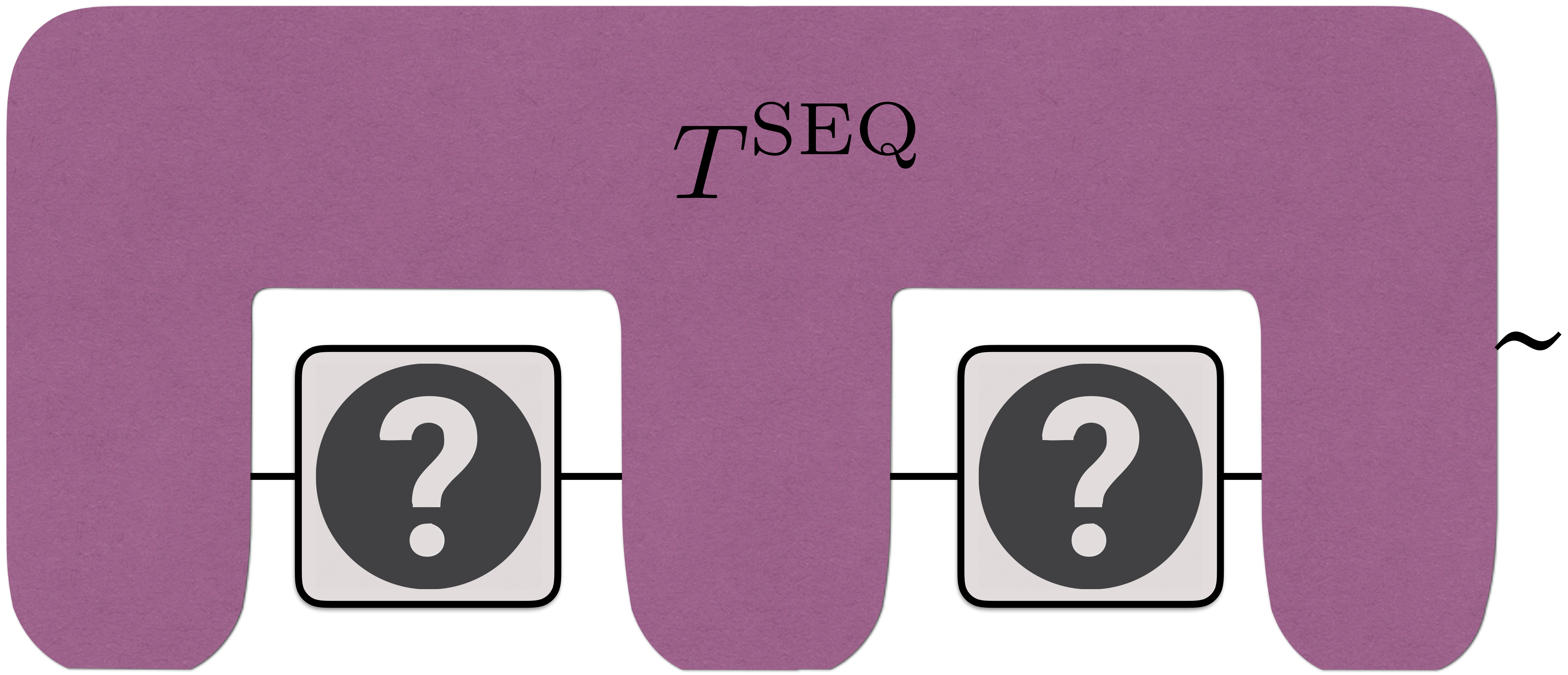
TWO COPIES



ρ
M
?
?
$\sim$
ρ
$\widetilde{E}$
?
?
M
$\sim$


T^{PAR}
$\sim$
T^{SEQ}
$\sim$



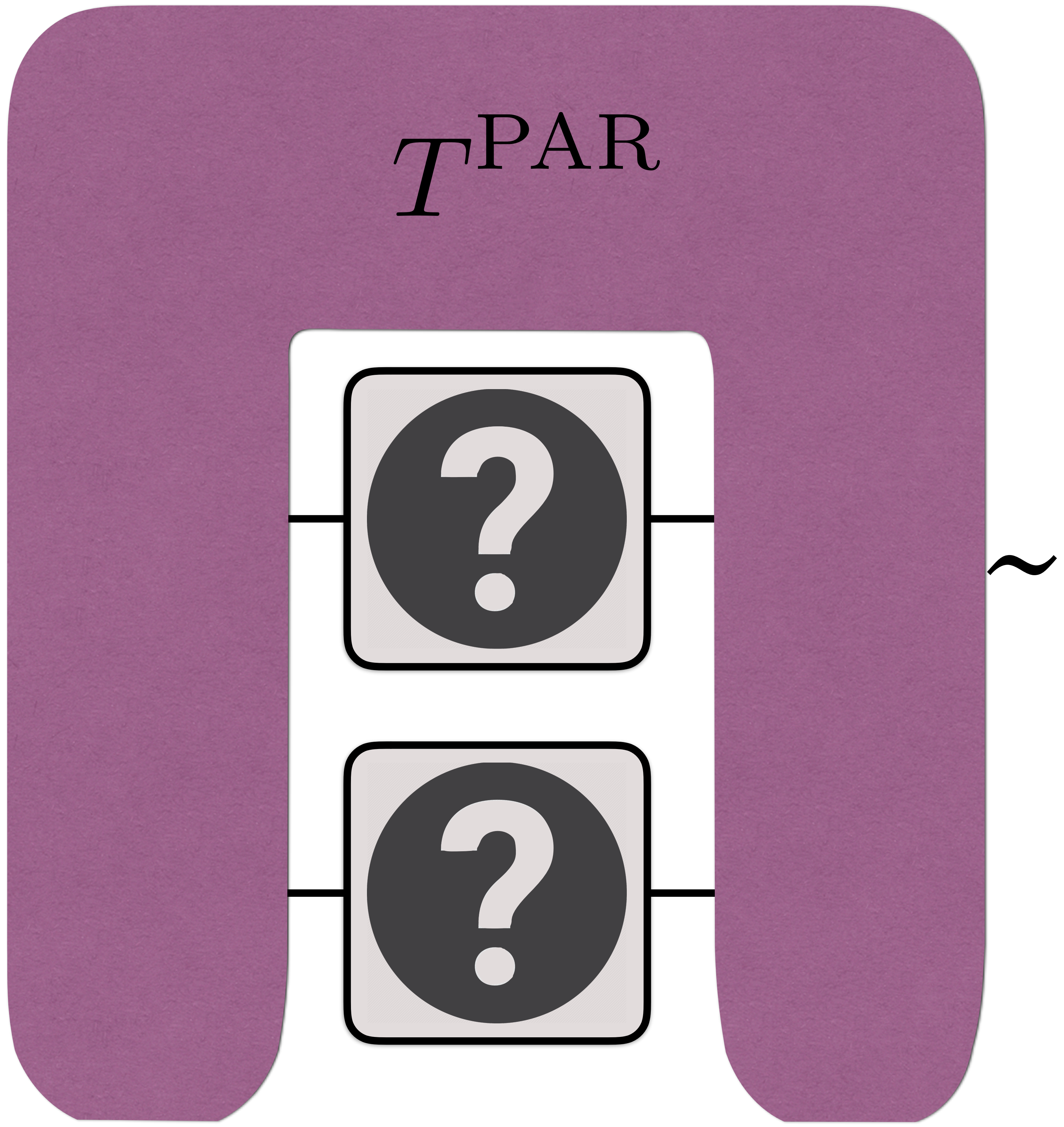


PARALLEL

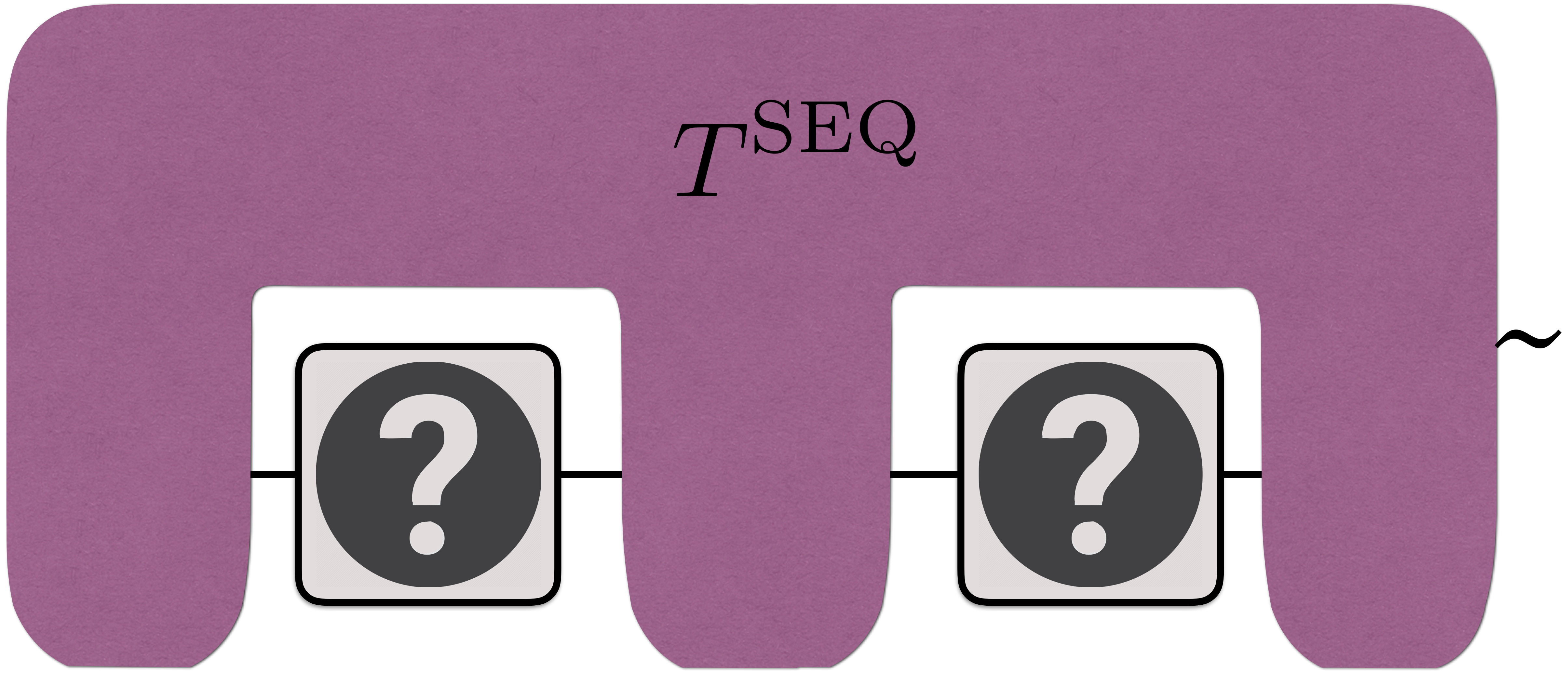
$\rho, \{M_i\}$ ⇄ $\{T_i^{\mathrm{PAR}}\}$

SEQUENTIAL

$\rho, \widetilde{E}, \{M_i\}$ ⇄ $\{T_i^{\mathrm{SEQ}}\}$

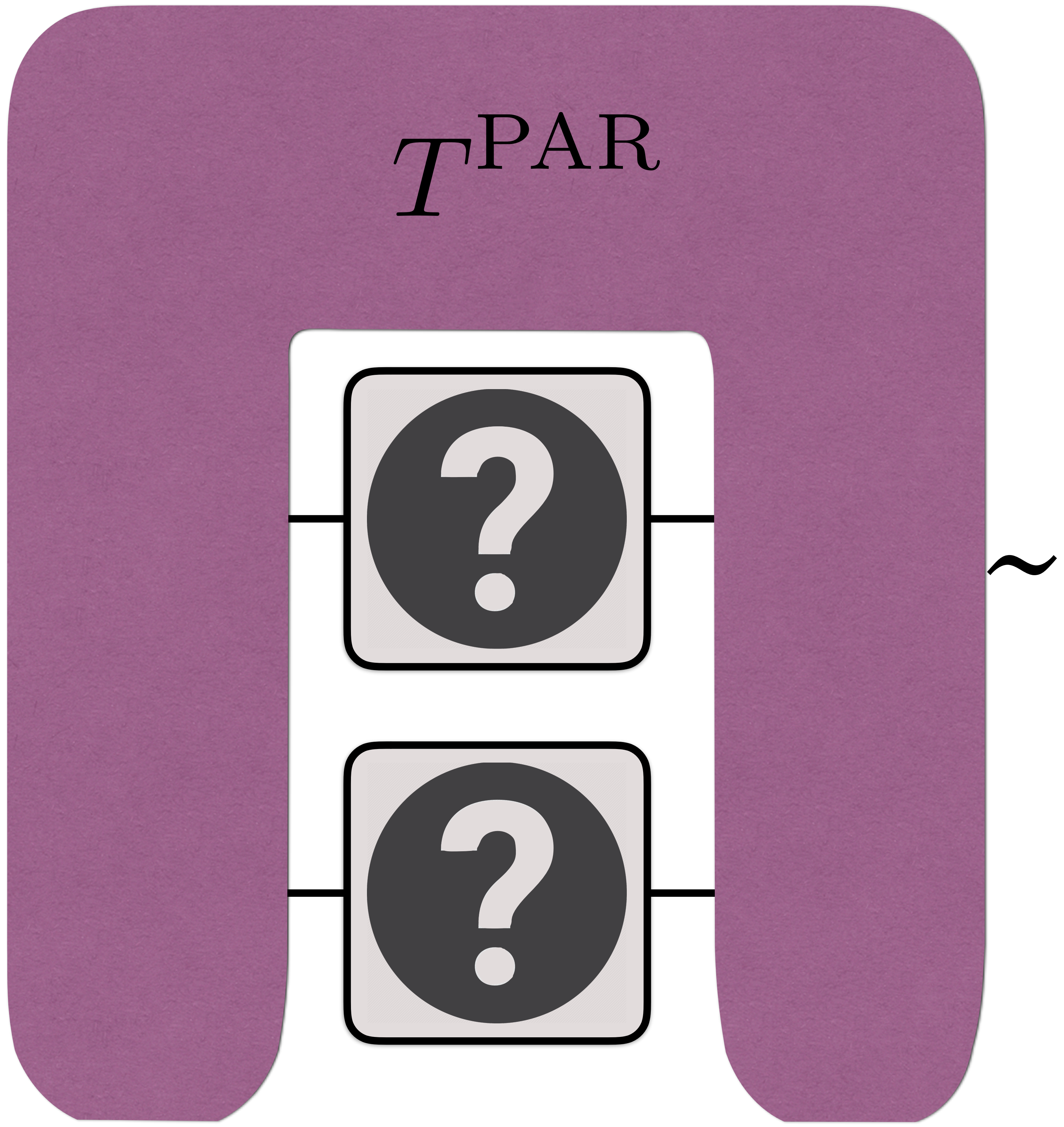


PARALLEL

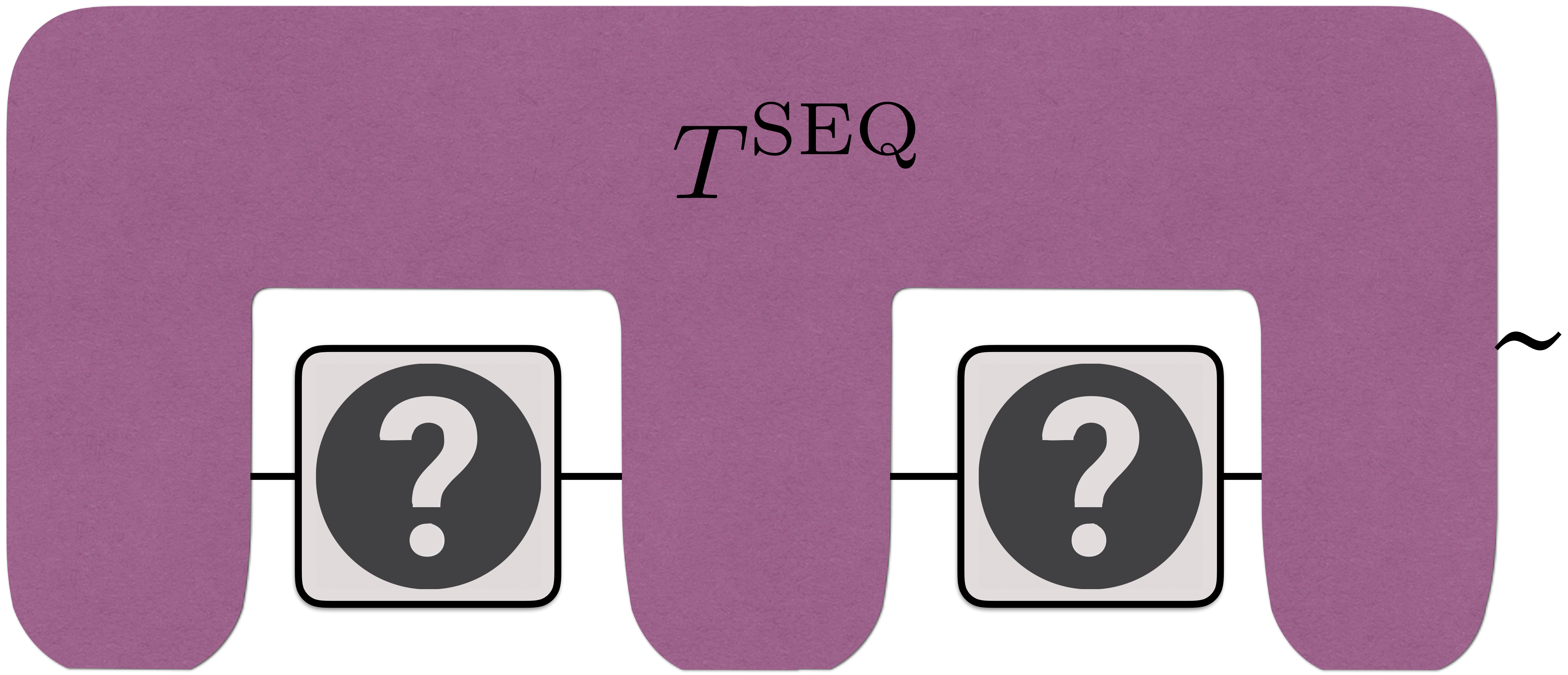


SEQUENTIAL

ALSO CHARACTERISED BY SDP CONSTRAINTS



PARALLEL



SEQUENTIAL

$$P^{\text{PAR}} = \max_{\{T_i^{\text{PAR}}\}} \sum_i p_i \, \mathrm{Tr}\left(C_i^{\otimes 2} \, T_i^{\text{PAR}}\right)$$

$$P^{\text{SEQ}} = \max_{\{T_i^{\text{SEQ}}\}} \sum_i p_i \, \mathrm{Tr}\left(C_i^{\otimes 2} \, T_i^{\text{SEQ}}\right)$$

GENERAL TESTERS

GENERAL TESTERS

- Extracting probability distributions from channels:

GENERAL TESTERS

- Extracting probability distributions from channels:

The most general bilinear function $f:(C_1, C_2) \to \mathbb{R}$ *that extracts valid probability distributions from a pair of Choi states of quantum channels* $C_1 \in L(H^{I_1} \otimes H^{O_1})$ *and* $C_2 \in L(H^{I_2} \otimes H^{O_2})$ *is*

$$p(i|C_1, C_2) = \mathrm{Tr}[(C_1 \otimes C_2)\, T_i^{\mathrm{GEN}}],$$

where $T^{\mathrm{GEN}} = \{T_i^{\mathrm{GEN}}\},\ T_i^{\mathrm{GEN}} \in L(H^{I_1} \otimes H^{O_1} \otimes H^{I_2} \otimes H^{O_2})$

is a ***general tester.***

GENERAL TESTERS

$$T^{\mathrm{GEN}} = \{T_i^{\mathrm{GEN}}\} \quad :$$

$$T_i^{\mathrm{GEN}} \geq 0 \quad \forall\ i$$

$$\sum_i T_i^{\mathrm{GEN}} = W^{\mathrm{GEN}}$$

GENERAL TESTERS

$$T^{\mathrm{GEN}} = \{T_i^{\mathrm{GEN}}\} \ :$$

$$T_i^{\mathrm{GEN}} \geq 0 \quad \forall\, i$$

$$\sum_i T_i^{\mathrm{GEN}} = W^{\mathrm{GEN}}$$

PROCESS MATRIX

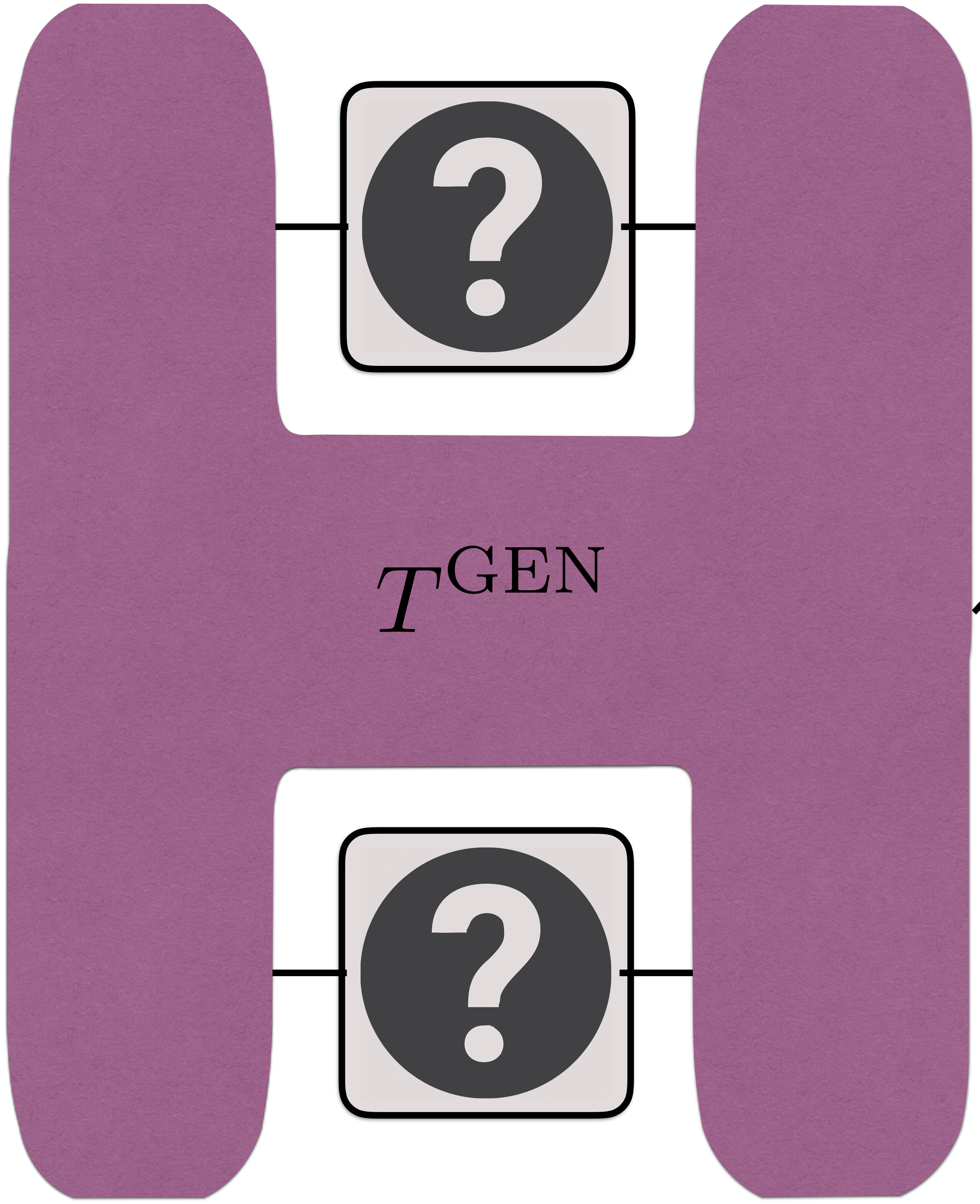
$$W^{\mathrm{GEN}} \geq 0$$

$$\mathrm{Tr}[W^{\mathrm{GEN}}(C_1 \otimes C_2)] = 1$$

$$\forall\, C_1, C_2 \in \mathrm{CPTP}$$

O. Oreshkov, F. Costa, and Č. Brukner, Nat. Commun. 3, 1092 (2012), arXiv:1105.4464 [quant-ph].

GENERAL TESTERS



$\sim$

$$T^{\mathrm{GEN}} = \{T_i^{\mathrm{GEN}}\} \ :$$

$$T_i^{\mathrm{GEN}} \geq 0 \quad \forall\, i$$

$$\sum_i T_i^{\mathrm{GEN}} = W^{\mathrm{GEN}}$$

PROCESS MATRIX

$$W^{\mathrm{GEN}} \geq 0$$

$$\mathrm{Tr}[W^{\mathrm{GEN}}(C_1 \otimes C_2)] = 1$$

$$\forall\, C_1, C_2 \in \mathrm{CPTP}$$

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MAXIMAL PROBABILITY OF SUCCESS

$$P^{\mathcal{S}} = \max_{\{T_i^{\mathcal{S}}\}} \sum_i p_i \operatorname{Tr}\left(C_i^{\otimes k} T_i^{\mathcal{S}}\right)$$

SEMIDEFINITE PROGRAMMING (SDP) & COMPUTER-ASSISTED PROOFS

$$P^{\mathrm{PAR}} \leq P^{\mathrm{SEQ}} \leq P^{\mathrm{GEN}}$$

$$P^{\mathrm{PAR}} < P^{\mathrm{SEQ}} < P^{\mathrm{GEN}}$$

STRICT HIERARCHY BETWEEN DISCRIMINATION STRATEGIES

FOR BOTH ENSEMBLES OF UNITARY AND NON-UNITARY CHANNELS

UNITARY CHANNELS

LITERATURE

LITERATURE

UNITARY CHANNELS:

- for N=2 candidates and any finite number of copies k, $P^{\mathrm{PAR}} = P^{\mathrm{SEQ}}$ [1].
- for a set of N unitaries that form a group, and any finite number of copies k, $P^{\mathrm{PAR}} = P^{\mathrm{SEQ}}$ [1].
- no known advantage of sequential strategies.

[1] G. Chiribella, G. M. D'Ariano, and P. Perinotti, PRL 101, 180501 (2008), arXiv:0803.3237 [quant-ph]

RESULT 1

- ensemble: $\{p_i, U_i\}$,
- number of candidates: N
- number of copies: k

RESULT 1

- ensemble: $\{p_i, U_i\}$, $\{p_i\} \to$ **UNIFORM**
- number of candidates: N
- number of copies: k

RESULT 1

- ensemble: $\{p_i, U_i\}$, $\{p_i\} \rightarrow$ **UNIFORM** & $\{U_i\} \rightarrow$ **GROUP**
- number of candidates: N
- number of copies: k

RESULT 1

- ensemble: $\{p_i, U_i\}$,
- number of candidates: N
- number of copies: k

$\{p_i\} \rightarrow$ **UNIFORM** & $\{U_i\} \rightarrow$ **GROUP**

$$P^{\mathrm{PAR}} = P^{\mathrm{SEQ}}$$

[1] G. Chiribella, G. M. D'Ariano, and P. Perinotti, PRL 101, 180501 (2008), arXiv:0803.3237 [quant-ph]

RESULT 1: PARALLEL OPTIMALITY

- ensemble: $\{p_i, U_i\}$,
- number of candidates: N
- number of copies: k

$\{p_i\} \to$ **UNIFORM** & $\{U_i\} \to$ **GROUP**

$$P^{\mathrm{PAR}} = P^{\mathrm{SEQ}} = P^{\mathrm{GEN}}$$

RESULT 1: PARALLEL OPTIMALITY

- ensemble: $\{p_i, U_i\}$,
- number of candidates: N
- number of copies: k

$\{p_i\} \to$ **UNIFORM** & $\{U_i\} \to$ **GROUP**

$$P^{\mathrm{PAR}} = P^{\mathrm{SEQ}} = P^{\mathrm{GEN}}$$

SKETCH OF PROOF: explicit construction of parallel tester that attains the same probability of success as each general tester, when applied to a unitary group.

RESULT 2: ADVANTAGES OF SEQUENTIAL STRATEGIES

- ensemble: $\{p_i, U_i\}$, $\{p_i\} \to$ **UNIFORM** & $\{U_i\} \to$ ~~**GROUP**~~
- number of candidates: 4
- number of copies: 2

RESULT 2: ADVANTAGES OF SEQUENTIAL STRATEGIES

- ensemble: $\{p_i, U_i\}$,
- number of candidates: 4
- number of copies: 2

$\{p_i\} \to$ **UNIFORM** & $\{U_i\} \to$ ~~**GROUP**~~

$$P^{\mathrm{PAR}} < P^{\mathrm{SEQ}} = 1$$

$$\{p_i\} = \left\{\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right\}$$

$$\{U_i\} = \{\mathbb{I}, \sqrt{\sigma_X}, \sqrt{\sigma_Y}, \sqrt{\sigma_Z}\}$$

RESULT 2: ADVANTAGES OF SEQUENTIAL STRATEGIES

- ensemble: $\{p_i, U_i\}$,
- number of candidates: 4
- number of copies: 2

$\{p_i\} \rightarrow$ **UNIFORM** & $\{U_i\} \rightarrow$ ~~**GROUP**~~

$$P^{\mathrm{PAR}} < P^{\mathrm{SEQ}} = 1$$

COMPUTER ASSISTED PROOF

$$\{p_i\} = \left\{\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right\}$$

$$\{U_i\} = \{\mathbb{I}, \sqrt{\sigma_X}, \sqrt{\sigma_Y}, \sqrt{\sigma_Z}\}$$

RESULT 2: ADVANTAGES OF SEQUENTIAL STRATEGIES

- ensemble: $\{p_i, U_i\}$, $\{p_i\} \to$ ~~UNIFORM~~ & $\{U_i\} \to$ GROUP
- number of candidates: 8
- number of copies: 2

RESULT 2: ADVANTAGES OF SEQUENTIAL STRATEGIES

- ensemble: $\{p_i, U_i\}$,
- number of candidates: 8
- number of copies: 2

$\{p_i\} \rightarrow$ ~~UNIFORM~~ & $\{U_i\} \rightarrow$ GROUP

$$P^{\mathrm{PAR}} < P^{\mathrm{SEQ}}$$

$$\{p_i\} = \left\{ \frac{3}{31}, \frac{1}{31}, \frac{4}{31}, \frac{1}{31}, \frac{5}{31}, \frac{9}{31}, \frac{2}{31}, \frac{6}{31} \right\}$$

$$\{U_i\} = \{\mathbb{I}, \sigma_X, \sigma_Y, \sigma_Z, H, \sigma_X H, \sigma_Y H, \sigma_Z H\}$$

RESULT 2: ADVANTAGES OF SEQUENTIAL STRATEGIES

- ensemble: $\{p_i, U_i\}$,
- number of candidates: 8
- number of copies: 2

$\{p_i\} \to$ ~~UNIFORM~~ & $\{U_i\} \to$ GROUP

$$P^{\mathrm{PAR}} < P^{\mathrm{SEQ}}$$

COMPUTER ASSISTED PROOF

$$\{p_i\} = \left\{\frac{3}{31}, \frac{1}{31}, \frac{4}{31}, \frac{1}{31}, \frac{5}{31}, \frac{9}{31}, \frac{2}{31}, \frac{6}{31}\right\}$$

$$\{U_i\} = \{\mathbb{I}, \sigma_X, \sigma_Y, \sigma_Z, H, \sigma_X H, \sigma_Y H, \sigma_Z H\}$$

RESULT 3: ADVANTAGES OF GENERAL STRATEGIES

- ensemble: $\{p_i, U_i\}$, $\{p_i\} \rightarrow$ UNIFORM & $\{U_i\} \rightarrow$ ~~GROUP~~
- number of candidates: 4
- number of copies: **3**

RESULT 3: ADVANTAGES OF GENERAL STRATEGIES

- ensemble: $\{p_i, U_i\}$,
- number of candidates: 4
- number of copies: 3

$\{p_i\} \rightarrow$ UNIFORM & $\{U_i\} \rightarrow$ ~~GROUP~~

$$P^{\text{PAR}} < P^{\text{SEQ}} < P^{\text{GEN}}$$

$$H_y := |+_y\rangle\langle 0| + |-_y\rangle\langle 1|$$

$$|\pm_y\rangle := \frac{1}{\sqrt{2}}(|0\rangle \pm i|1\rangle)$$

$$H_P := |+_P\rangle\langle 0| + |-_P\rangle\langle 1|$$

$$|+_P\rangle := \frac{1}{5}(3|0\rangle + 4|1\rangle)$$

$$|-_P\rangle := \frac{1}{5}(4|0\rangle - 3|1\rangle)$$

$$\{p_i\} = \left\{\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right\}$$

$$\{U_i\} = \{\sqrt{\sigma_X}, \sqrt{\sigma_Z}, \sqrt{H_y}, \sqrt{H_P}\}$$

RESULT 3: ADVANTAGES OF GENERAL STRATEGIES

- ensemble: $\{p_i, U_i\}$,
- number of candidates: 4
- number of copies: 3

$\{p_i\} \rightarrow$ UNIFORM & $\{U_i\} \rightarrow$ ~~GROUP~~

$$P^{\text{PAR}} < P^{\text{SEQ}} < P^{\text{GEN}}$$

$$\{p_i\} = \left\{\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right\}$$

COMPUTER ASSISTED PROOF

$$\{U_i\} = \{\sqrt{\sigma_X}, \sqrt{\sigma_Z}, \sqrt{H_y}, \sqrt{H_P}\}$$

RESULT 3: ADVANTAGES OF GENERAL STRATEGIES

- ensemble: $\{p_i, U_i\}$, $\{p_i\} \rightarrow$ ~~UNIFORM~~ & $\{U_i\} \rightarrow$ GROUP
- number of candidates: ∞
- number of copies: 3

RESULT 3: ADVANTAGES OF GENERAL STRATEGIES

- ensemble: $\{p_i, U_i\}$,
- number of candidates: ∞
- number of copies: 3

$\{p_i\} \rightarrow$ ~~UNIFORM~~ & $\{U_i\} \rightarrow$ GROUP

$$P^{\mathrm{PAR}} < P^{\mathrm{SEQ}} < P^{\mathrm{GEN}}$$

$$\{p_i\} = \left\{ \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0, 0, 0, \ \ldots \right\}$$

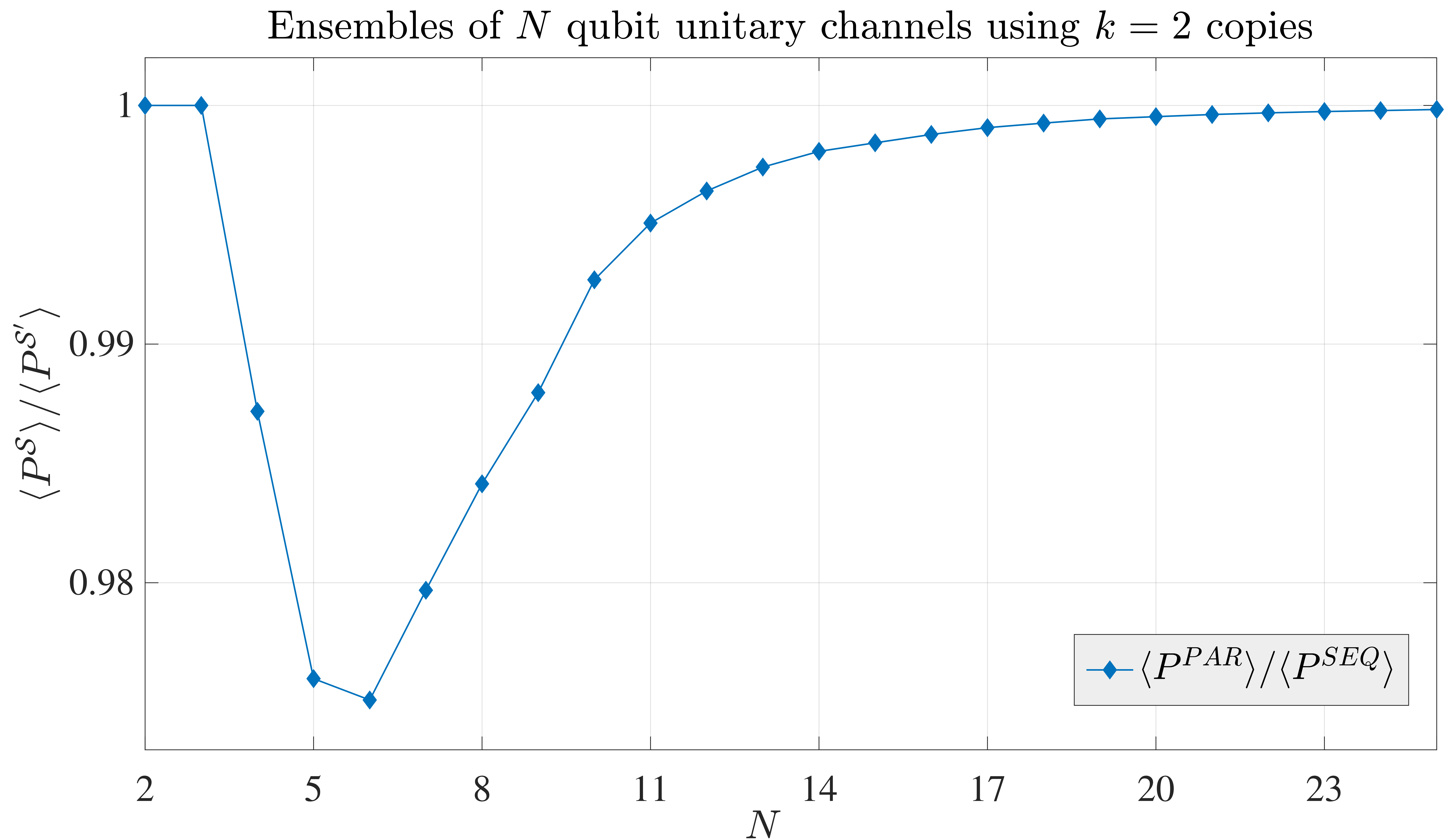
$$\{U_i\} = \{\sqrt{\sigma_X}, \sqrt{\sigma_Z}, \sqrt{H_Y}, \sqrt{H_P}, \ \ldots\}$$

Uniformly sampling qubit unitary channels

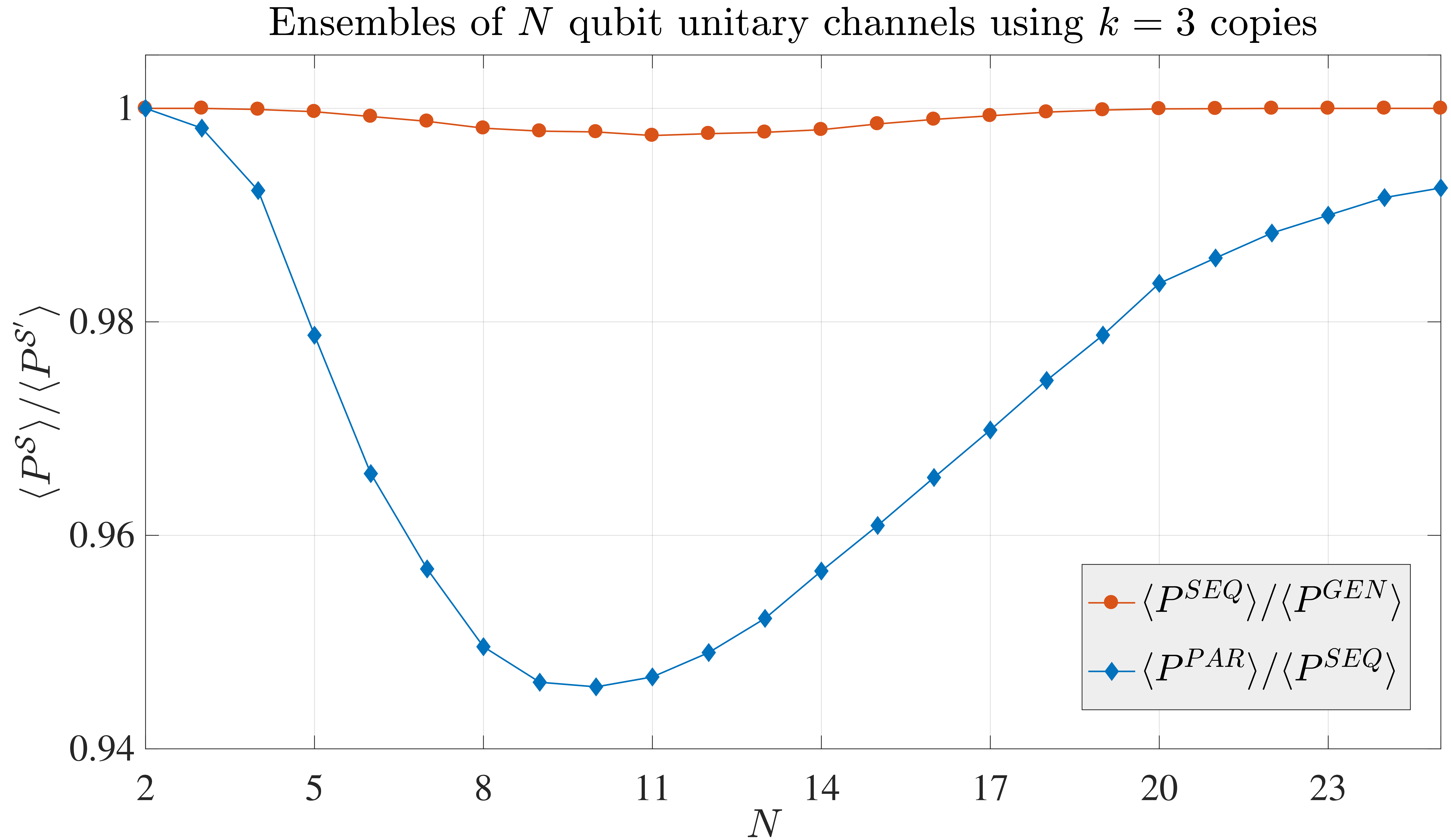
N	$k=2$	$k=3$
2	$\boldsymbol{P^{\mathrm{PAR}} = P^{\mathrm{SEQ}}} = P^{\mathrm{GEN}}$	$\boldsymbol{P^{\mathrm{PAR}} = P^{\mathrm{SEQ}}} = P^{\mathrm{GEN}}$
3	$P^{\mathrm{PAR}} = P^{\mathrm{SEQ}} = P^{\mathrm{GEN}}$	$P^{\mathrm{PAR}} < P^{\mathrm{SEQ}} = P^{\mathrm{GEN}}$
4	$P^{\mathrm{PAR}} < P^{\mathrm{SEQ}} = P^{\mathrm{GEN}}$	$P^{\mathrm{PAR}} < P^{\mathrm{SEQ}} < P^{\mathrm{GEN}}$
$\vdots$	$\vdots$	$\vdots$
10	$P^{\mathrm{PAR}} < P^{\mathrm{SEQ}} = P^{\mathrm{GEN}}$	$P^{\mathrm{PAR}} < P^{\mathrm{SEQ}} < P^{\mathrm{GEN}}$
$\vdots$	$\vdots$	$\vdots$
25	$P^{\mathrm{PAR}} \approx P^{\mathrm{SEQ}} = P^{\mathrm{GEN}}$	$P^{\mathrm{PAR}} < P^{\mathrm{SEQ}} \approx P^{\mathrm{GEN}}$

Uniformly sampling qubit unitary channels

N	$k=2$	$k=3$
2	$\boldsymbol{P^{\mathrm{PAR}} = P^{\mathrm{SEQ}}} = P^{\mathrm{GEN}}$	$\boldsymbol{P^{\mathrm{PAR}} = P^{\mathrm{SEQ}}} = P^{\mathrm{GEN}}$
3	$P^{\mathrm{PAR}} = P^{\mathrm{SEQ}} = P^{\mathrm{GEN}}$	$P^{\mathrm{PAR}} < P^{\mathrm{SEQ}} = P^{\mathrm{GEN}}$
4	$P^{\mathrm{PAR}} < P^{\mathrm{SEQ}} = P^{\mathrm{GEN}}$	$P^{\mathrm{PAR}} < P^{\mathrm{SEQ}} < P^{\mathrm{GEN}}$
$\vdots$	$\vdots$	$\vdots$
10	$P^{\mathrm{PAR}} < P^{\mathrm{SEQ}} = P^{\mathrm{GEN}}$	$P^{\mathrm{PAR}} < P^{\mathrm{SEQ}} < P^{\mathrm{GEN}}$
$\vdots$	$\vdots$	$\vdots$
25	$P^{\mathrm{PAR}} \approx P^{\mathrm{SEQ}} = P^{\mathrm{GEN}}$	$P^{\mathrm{PAR}} < P^{\mathrm{SEQ}} \approx P^{\mathrm{GEN}}$



Ensembles of N qubit unitary channels using $k = 2$ copies
$\langle P^S \rangle / \langle P^{S'} \rangle$
1
0.99
0.98
2
5
8
11
14
17
20
23
N
$\langle P^{PAR} \rangle / \langle P^{SEQ} \rangle$



Ensembles of N qubit unitary channels using $k = 3$ copies
$\langle P^{S} \rangle / \langle P^{S'} \rangle$
1
0.98
0.96
0.94
2
5
8
11
14
17
20
23
N
$\langle P^{SEQ} \rangle / \langle P^{GEN} \rangle$
$\langle P^{PAR} \rangle / \langle P^{SEQ} \rangle$

RESULT 4: ABSOLUTE UPPER BOUND

$$P^{GEN} \leq \frac{1}{N} \frac{(k + d^2 - 1)!}{k!(d^2 - 1)!}$$

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BOUND ATTAINED BY GROUPS OF UNITARIES THAT FORM A k-DESIGN

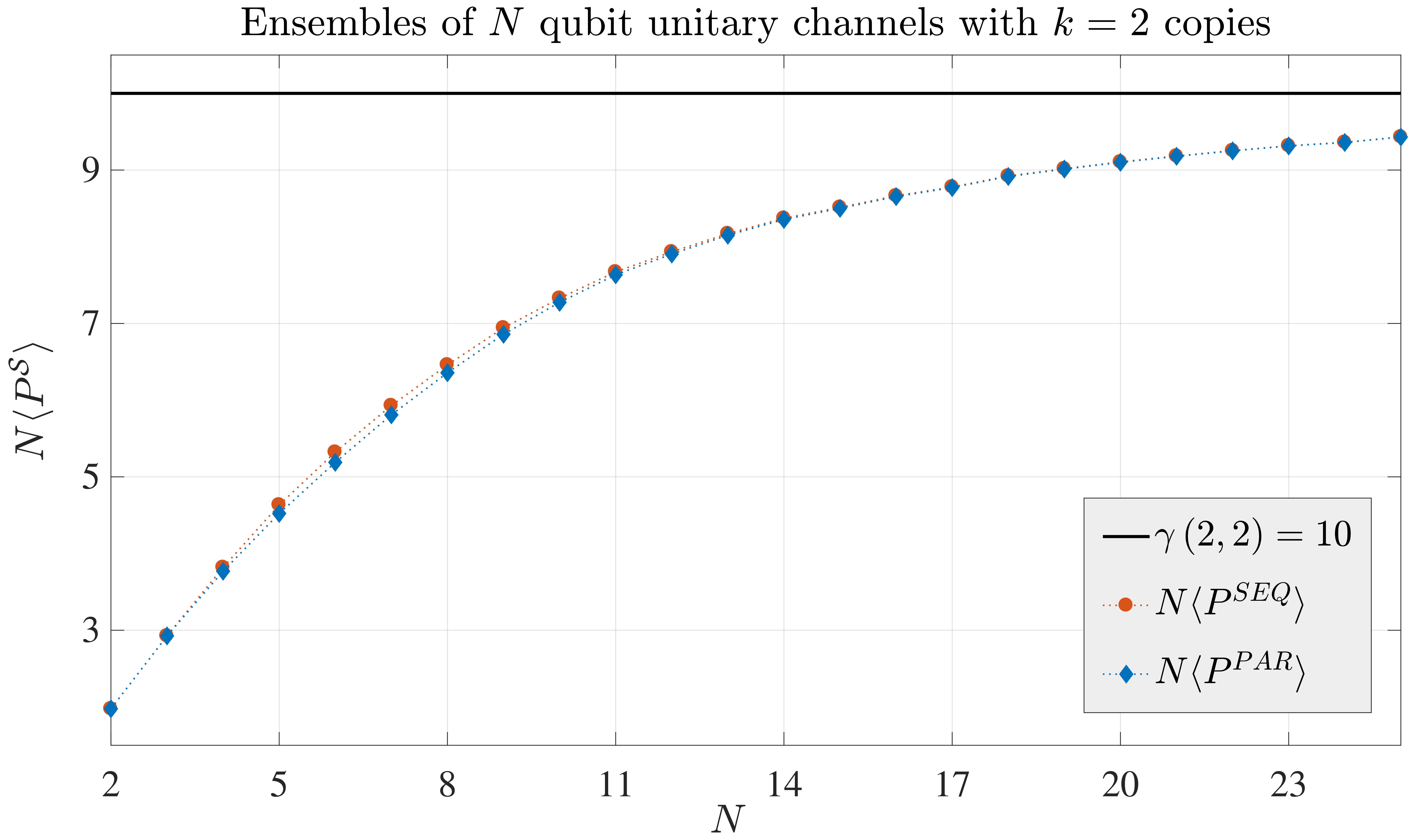
RESULT 4: ABSOLUTE UPPER BOUND

$$P^{GEN} \leq \frac{1}{N} \frac{(k + d^2 - 1)!}{k!(d^2 - 1)!}$$

SKETCH OF PROOF: analytically exhibit a feasible point in the dual problem of the maximal probability of discrimination attained by general strategies; this induces an upper bound for all strategies.

RESULT 4: ABSOLUTE UPPER BOUND

$$P^{GEN} \leq \frac{1}{N} \frac{(k+d^2-1)!}{k!(d^2-1)!} =: \frac{1}{N}\gamma(d,k)$$



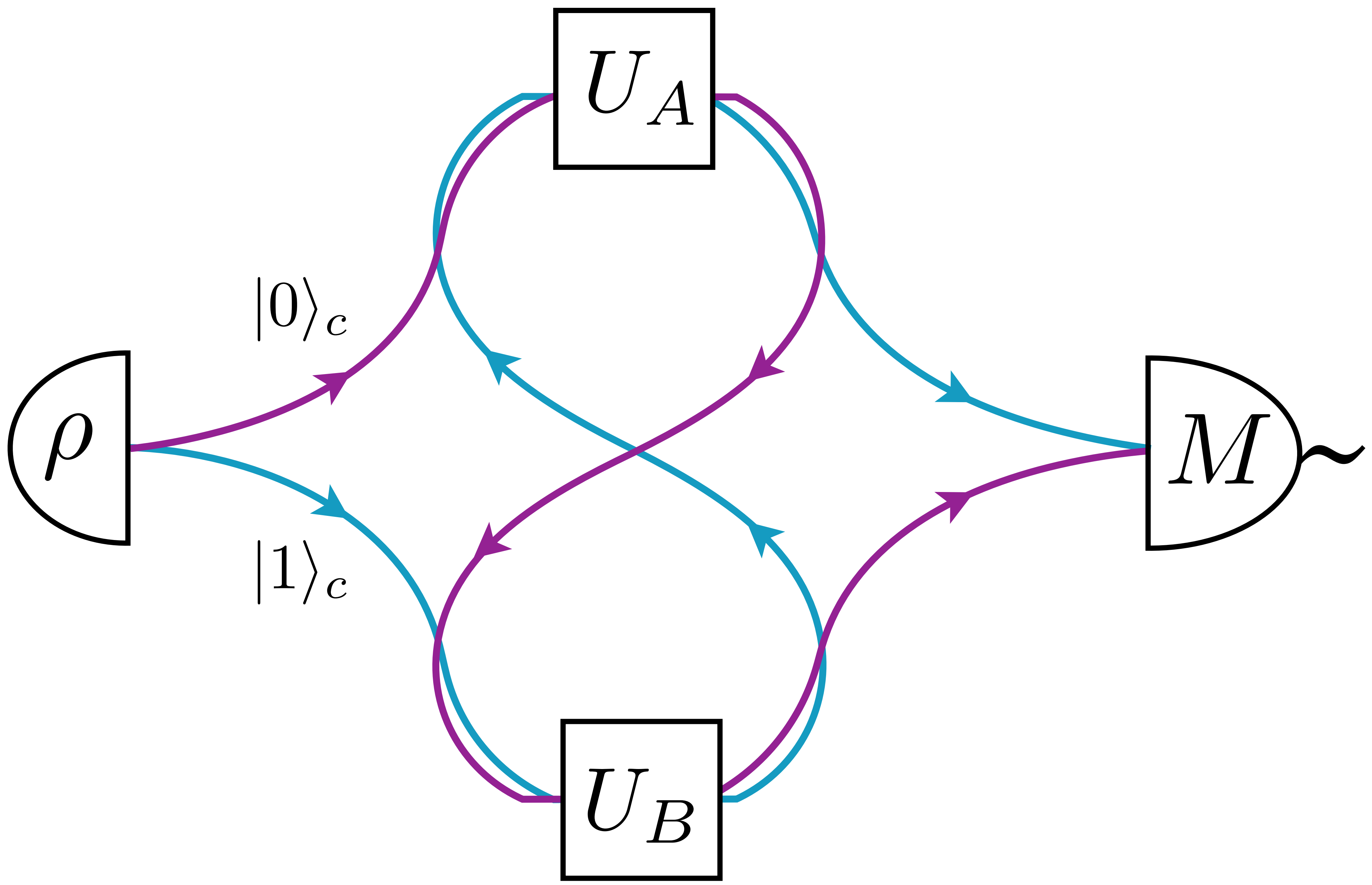
Ensembles of N qubit unitary channels with $k = 2$ copies
$N\langle P^{S}\rangle$
9
7
5
3
2
5
8
11
14
17
20
23
N
$\gamma(2,2) = 10$
$N\langle P^{SEQ}\rangle$
$N\langle P^{PAR}\rangle$

Ensembles of N qubit unitary channels with $k = 3$ copies
$N\langle P^{S}\rangle$
20
15
10
5
2
5
8
11
14
17
20
23
N
$\gamma(2,3) = 20$
$N\langle P^{GEN}\rangle$
$N\langle P^{SEQ}\rangle$
$N\langle P^{PAR}\rangle$

RESULT 5: NO ADVANTAGE OF SWITCH-LIKE STRATEGIES

- ensemble: $\{p_i, U_i\}$,
- number of candidates: N
- number of copies: k

QUANTUM SWITCH



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- ensemble: $\{p_i, U_i\}$,
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- number of copies: k

SWITCH-LIKE STRATEGIES

$$\mathcal{T}^{\mathrm{SEQ}} \subset \mathcal{T}^{\mathrm{SL}} \subset \mathcal{T}^{\mathrm{GEN}}$$

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SWITCH-LIKE STRATEGIES

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However, there are advantages for $k=2$ and non-unitary channels

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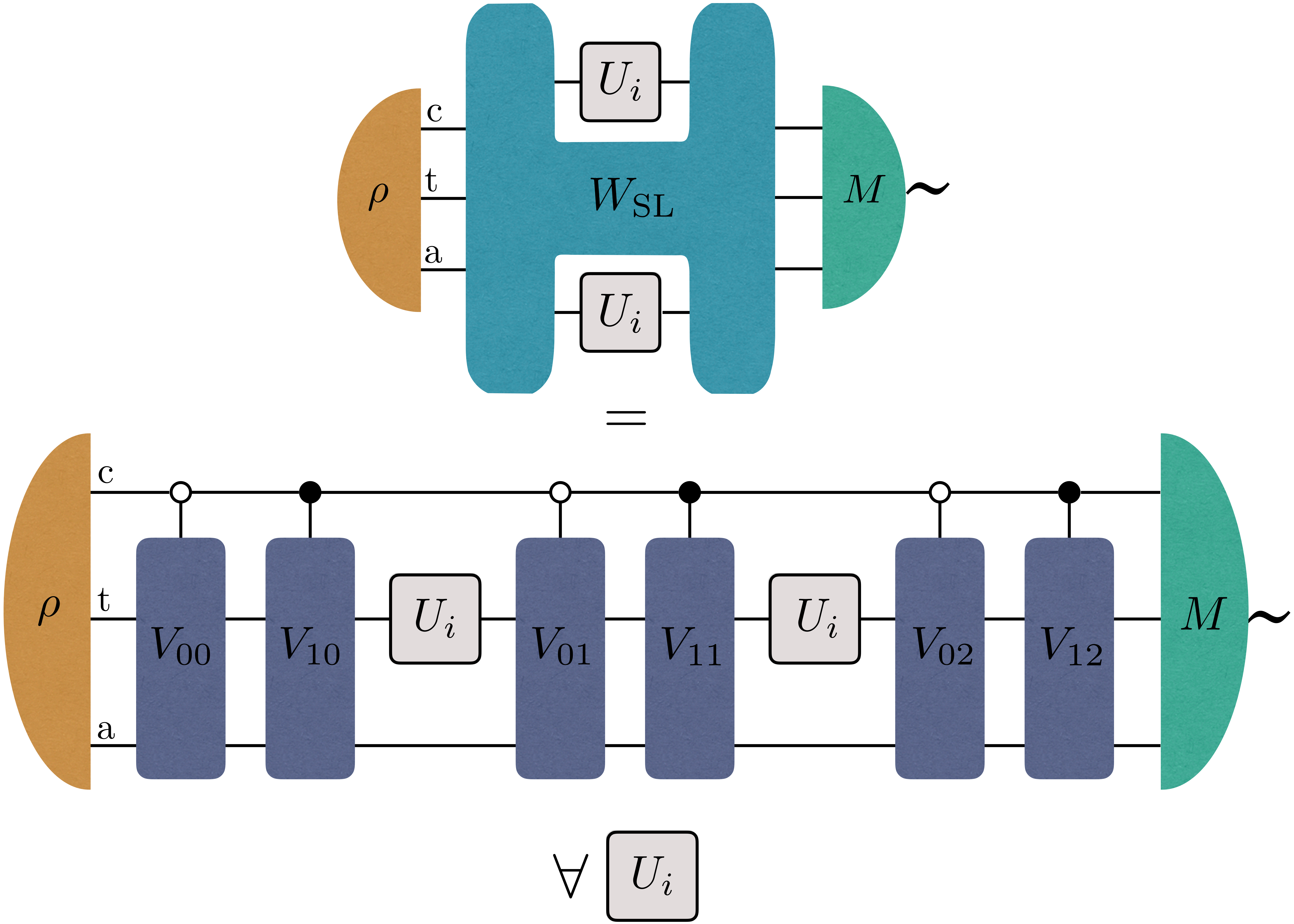
However, there are advantages for $k=2$ and non-unitary channels

SKETCH OF PROOF: exhibit sequential strategy that attains same probability of success



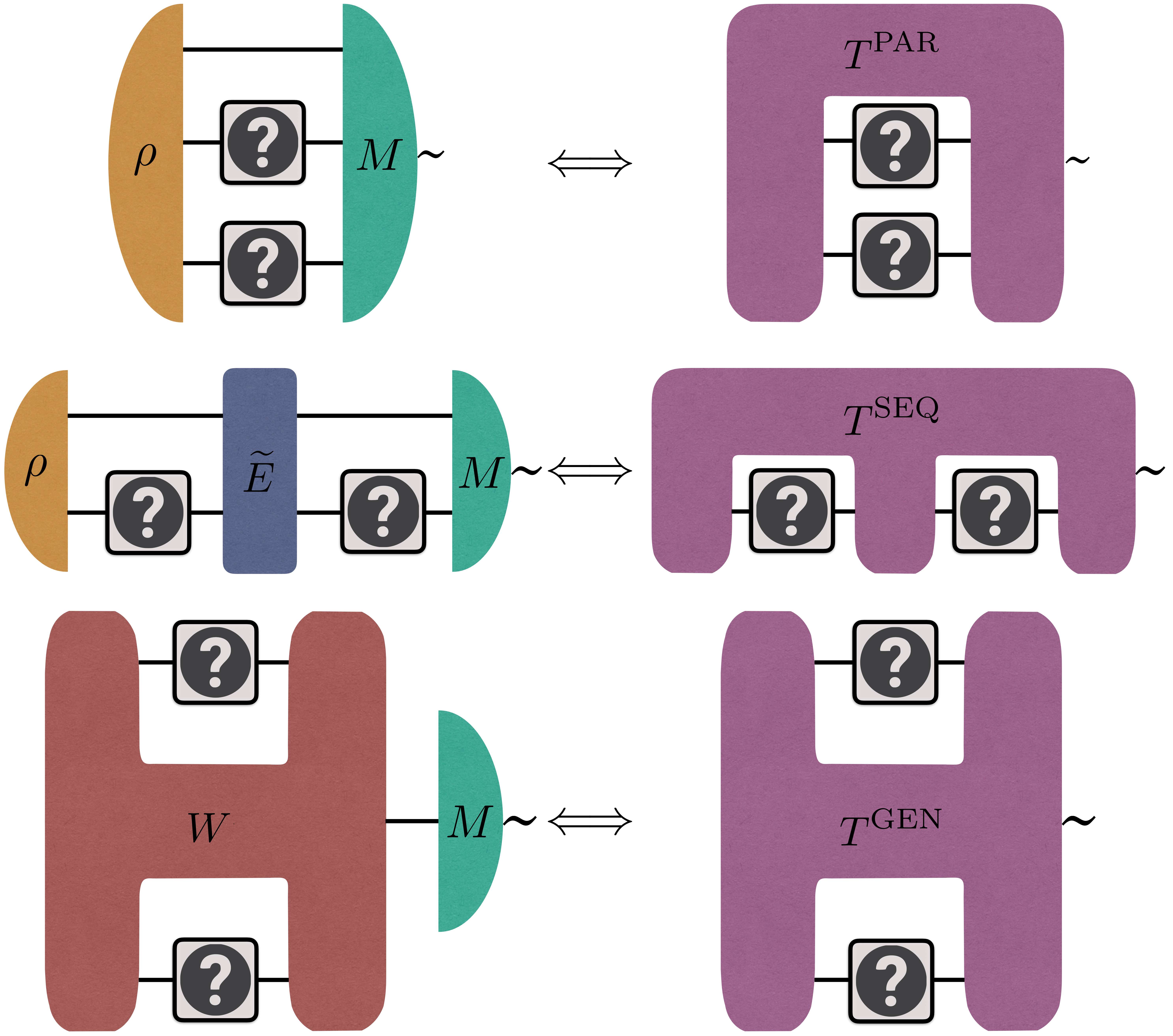
c
t
a
ρ
U_i
W_SL
U_i
M
=
c
t
a
ρ
V_00
V_10
U_i
V_01
V_11
U_i
V_02
V_12
M
∀ U_i

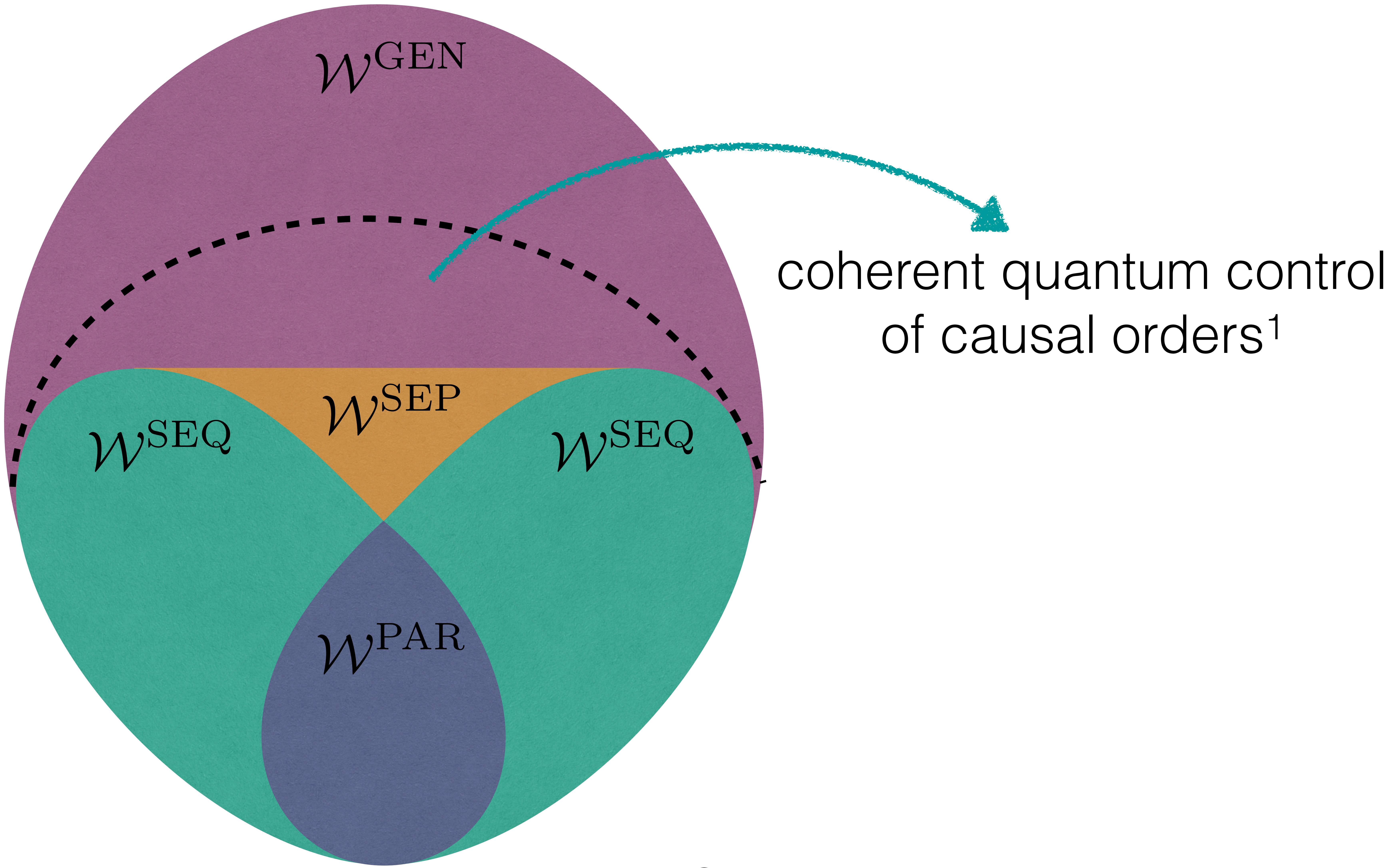
IMPLEMENTATION

PARALLEL

SEQUENTIAL

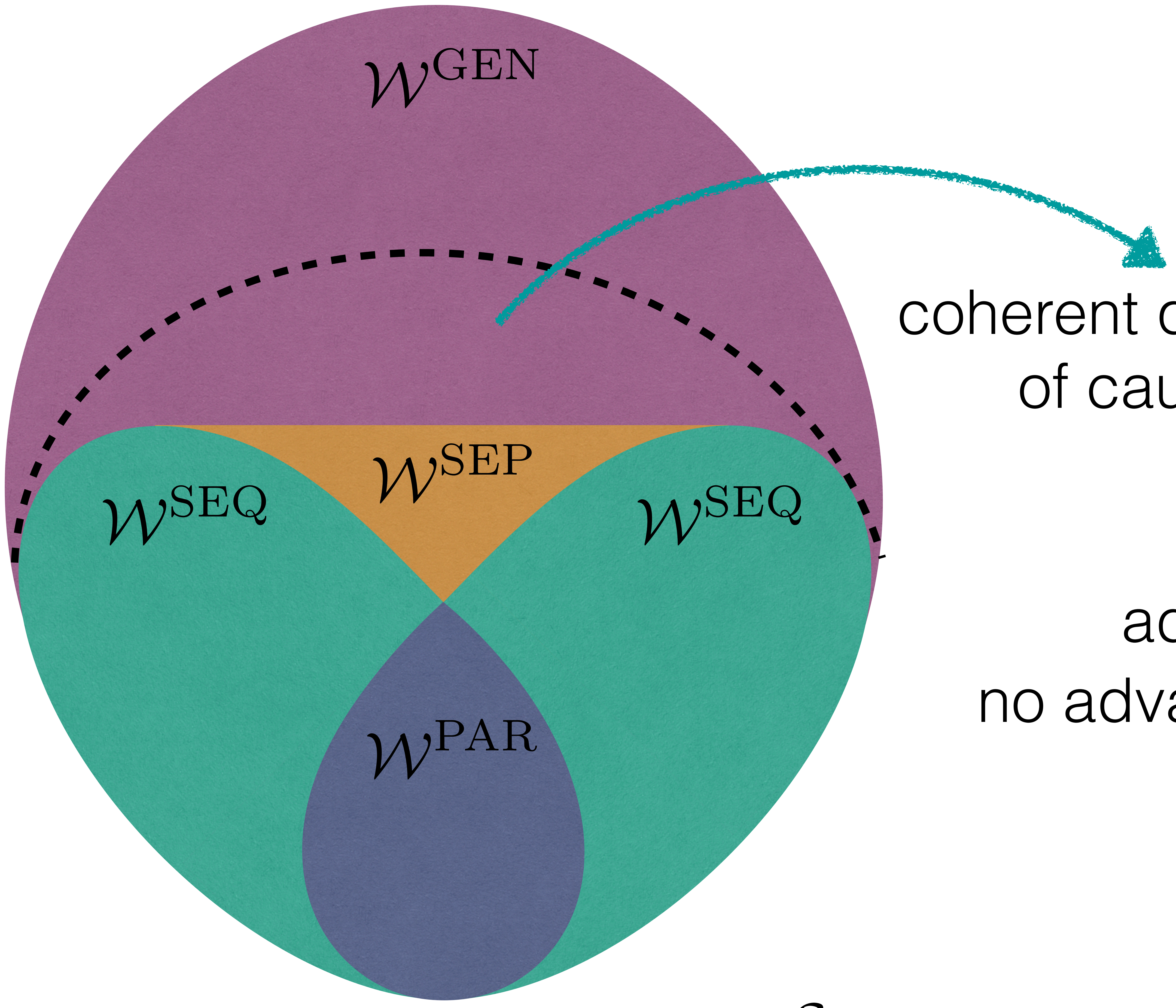
GENERAL





Set of all processes $\mathcal{W}^{\mathcal{S}}$

[1] J. Wechs, H. Dourdent, A. A. Abbott, C. Branciard, PRX Quantum 2, 030335 (2021), arXiv: 2101.08796 [quant-ph] (2021)



Set of all processes $\mathcal{W}^{\mathcal{S}}$

coherent quantum control of causal orders[1]

advantage for general channels

no advantage for unitaries with k=2 copies

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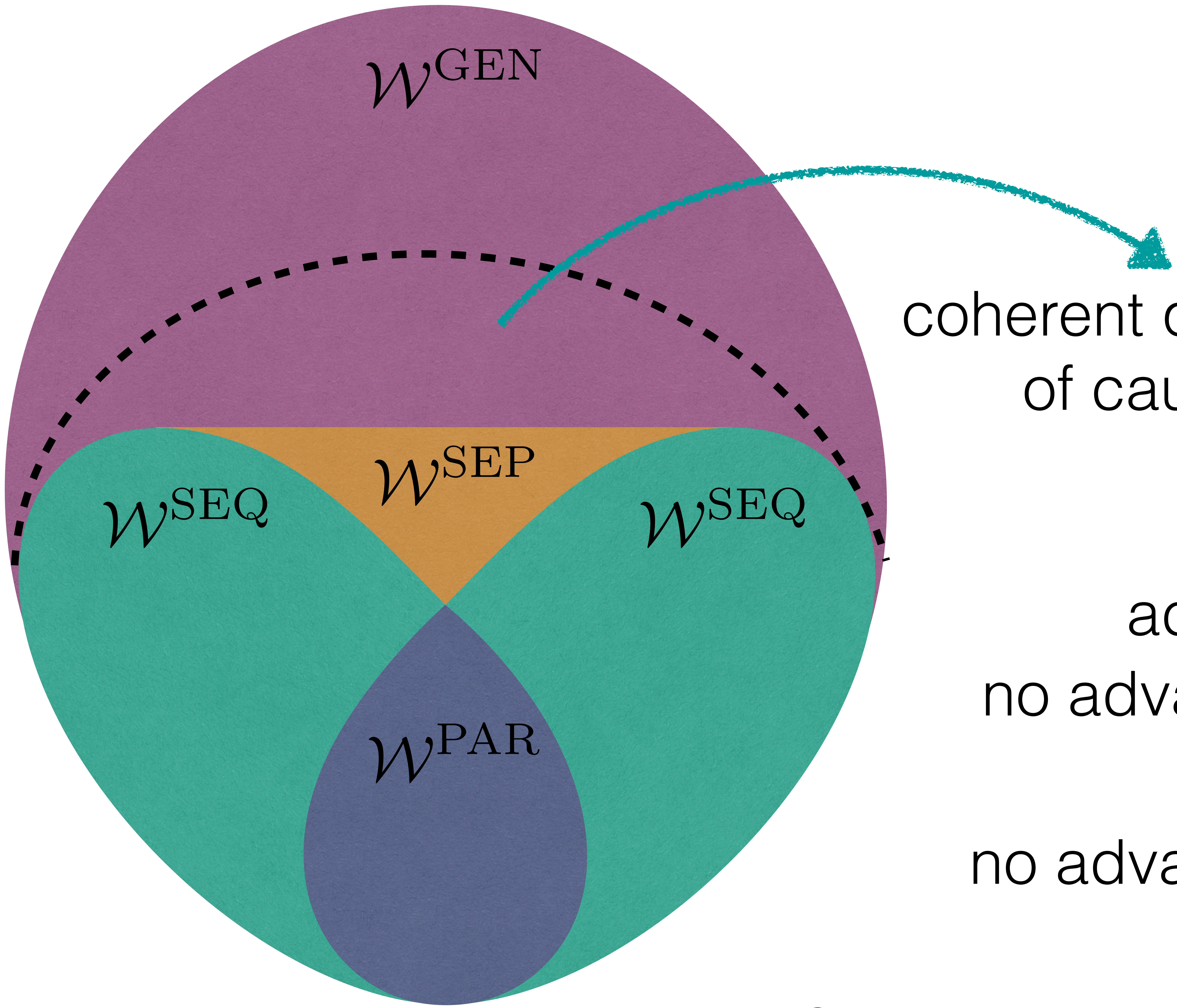


Set of all processes $\mathcal{W}^{\mathcal{S}}$

coherent quantum control of causal orders[1]

advantage for general channels
no advantage for unitaries with k=2 copies

no advantage for unitaries with k>2 copies[2]

[1] J. Wechs, H. Dourdent, A. A. Abbott, C. Branciard, PRX Quantum 2, 030335 (2021), arXiv: 2101.08796 [quant-ph] (2021)

[2] A. A. Abbott, M. Mhalla, P. Pocreau, arXiv: 2307.10285 [quant-ph] (2023)

CONCLUSIONS

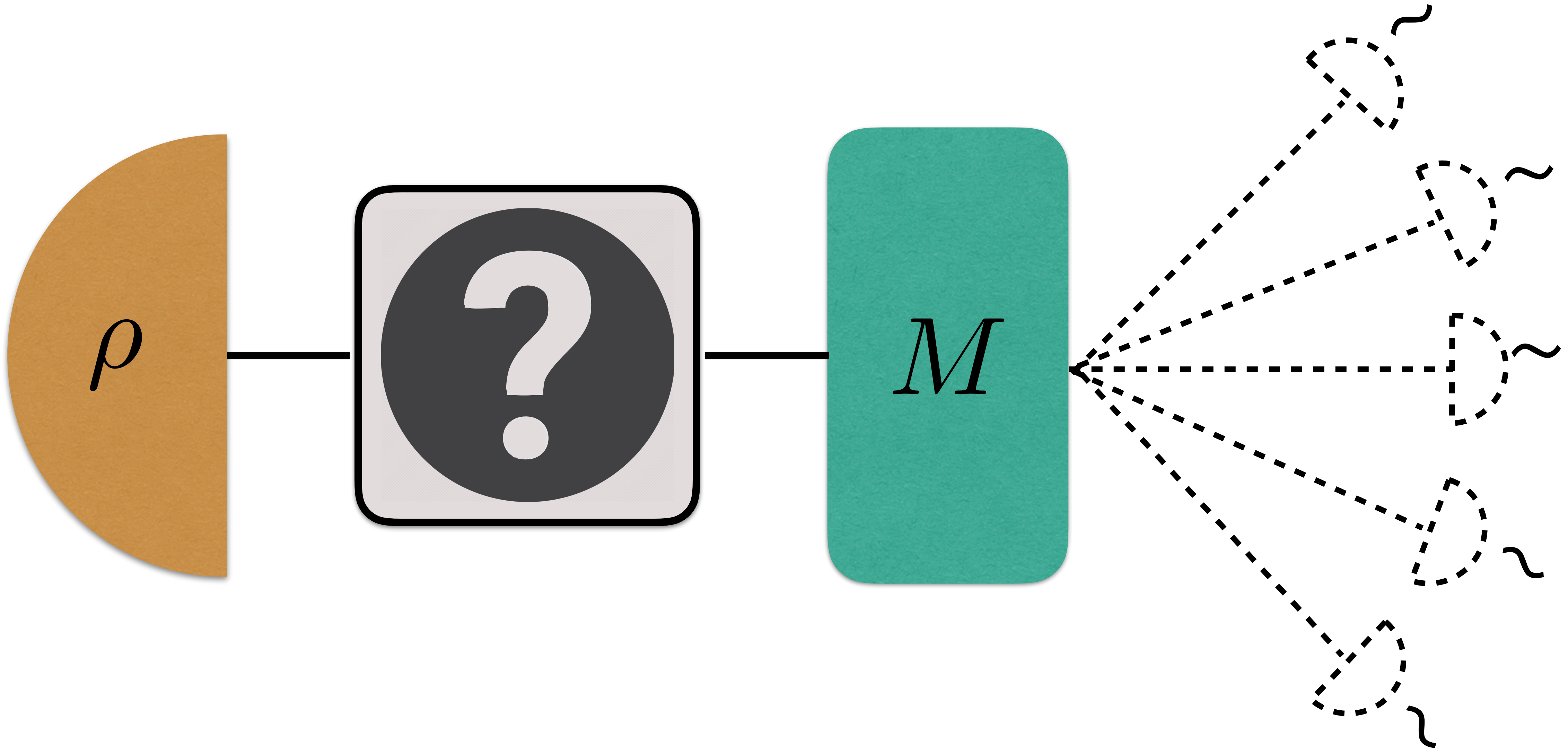
CONCLUSIONS

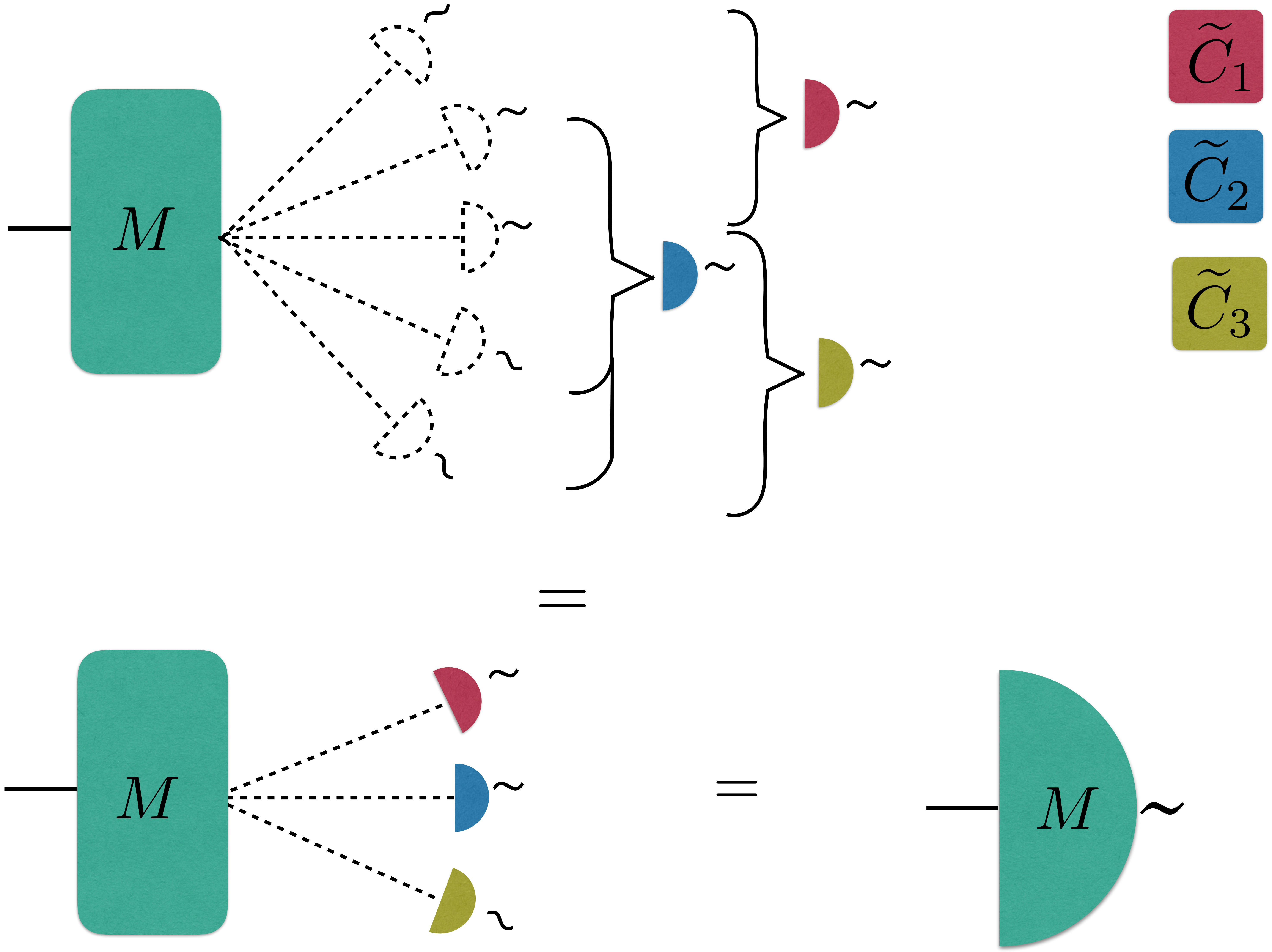
- **Unified tester formalism** that includes indefinite-causal-order strategies and method for **computer-assisted proofs** based on SDPs.
- **Parallel strategies are indeed optimal** for discrimination unitaries that form a **group** with **uniform** probability distribution.
- **Strict hierarchy** between discrimination strategies for unitary channels.
- **Absolute upper bound** for maximal probability of success with any conceivable strategy.
- **Switch-like strategies not useful** for N **unitaries** with k copies and any probability distribution.

THANK YOU!

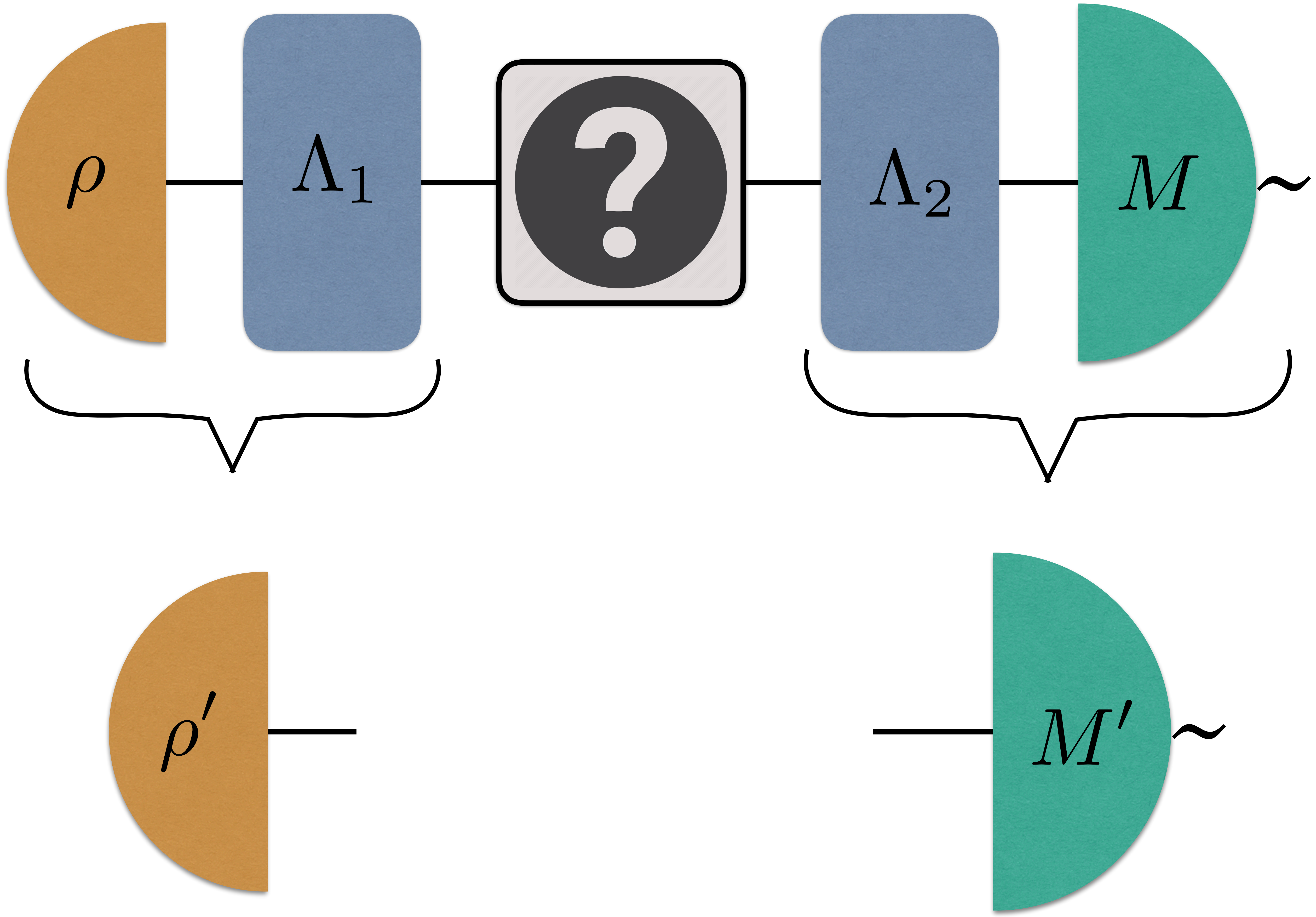
EXTRA

STRATEGY





(as many outcomes as candidates)



ρ
Λ_1
Λ_2
M
ρ'
M'

EXAMPLE

EXAMPLE

ENSEMBLE:

$$\{p_i\} = \left\{\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right\}$$

$$\{C_i\} = \{\mathbb{I}, \sigma_X, \sigma_Y, \sigma_Z\}$$

EXAMPLE

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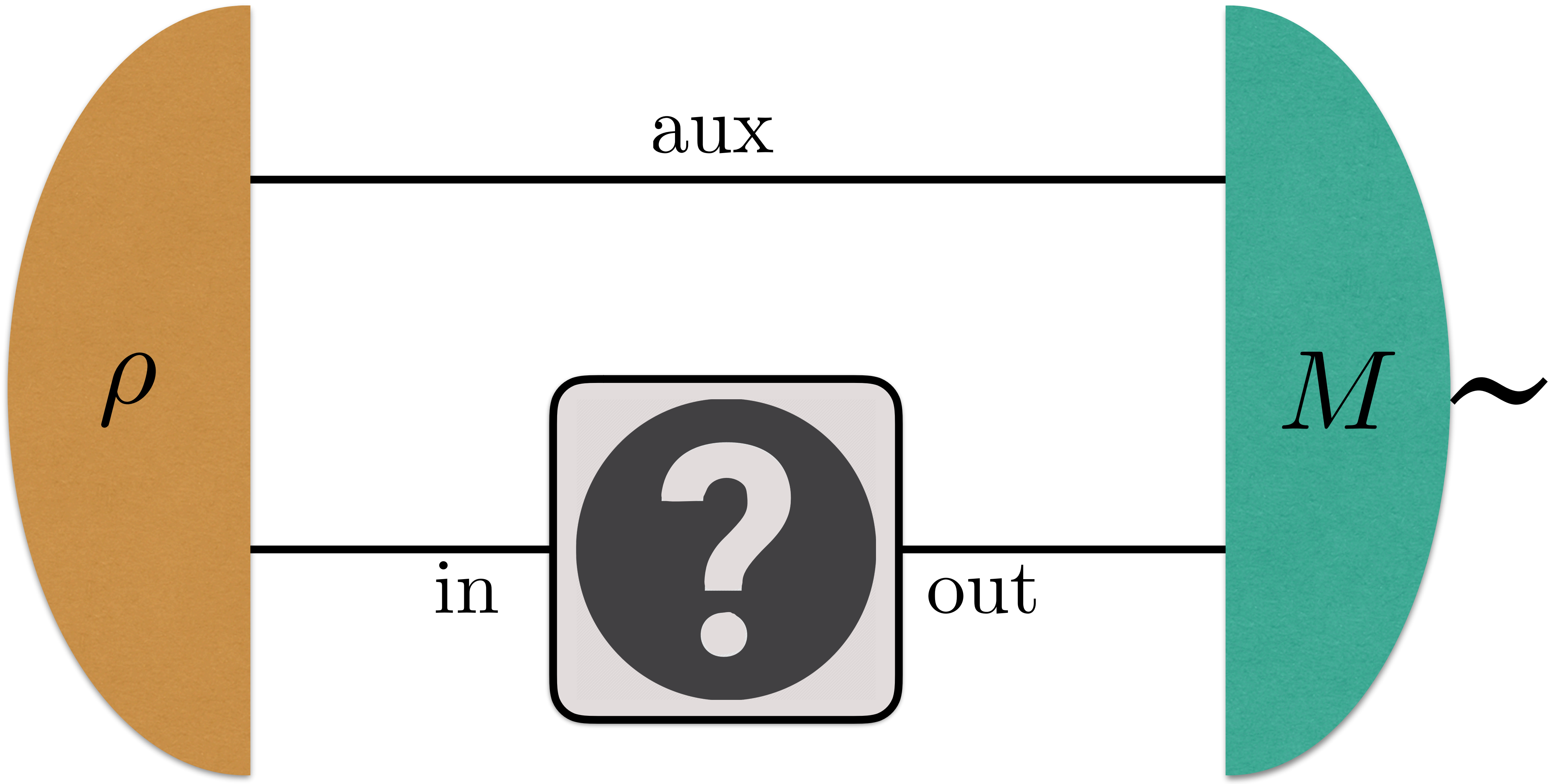
$$\{C_i\} = \{\mathbb{I}, \sigma_X, \sigma_Y, \sigma_Z\}$$



EXAMPLE

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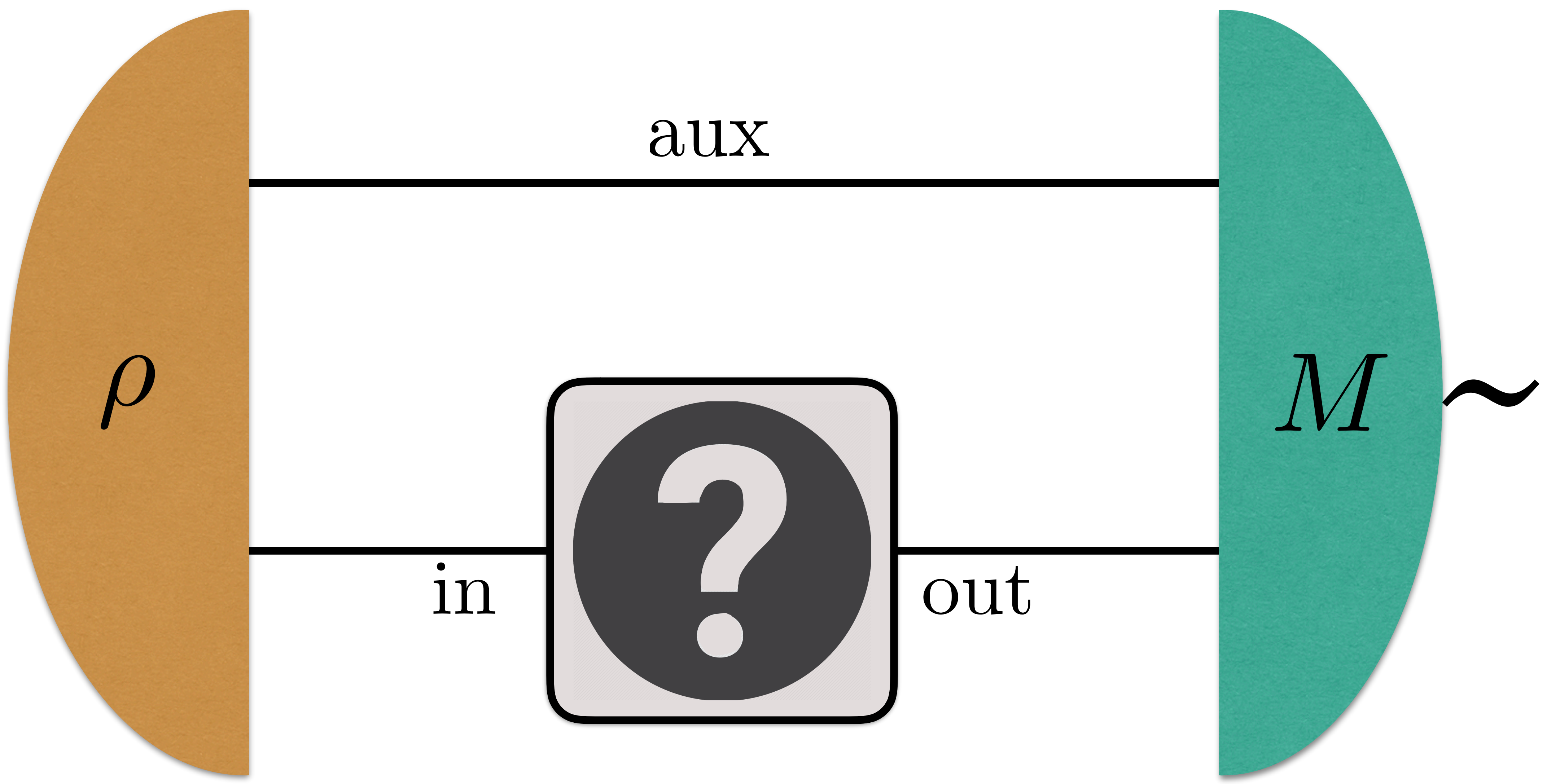

STRATEGY:

$$\rho = |\Phi^+\rangle\langle\Phi^+|$$

$$\{M_i\} = \{|\Phi^+\rangle\langle\Phi^+|, \\ |\Phi^-\rangle\langle\Phi^-|, \\ |\Psi^+\rangle\langle\Psi^+|, \\ |\Psi^-\rangle\langle\Psi^-|\}$$

EXAMPLE

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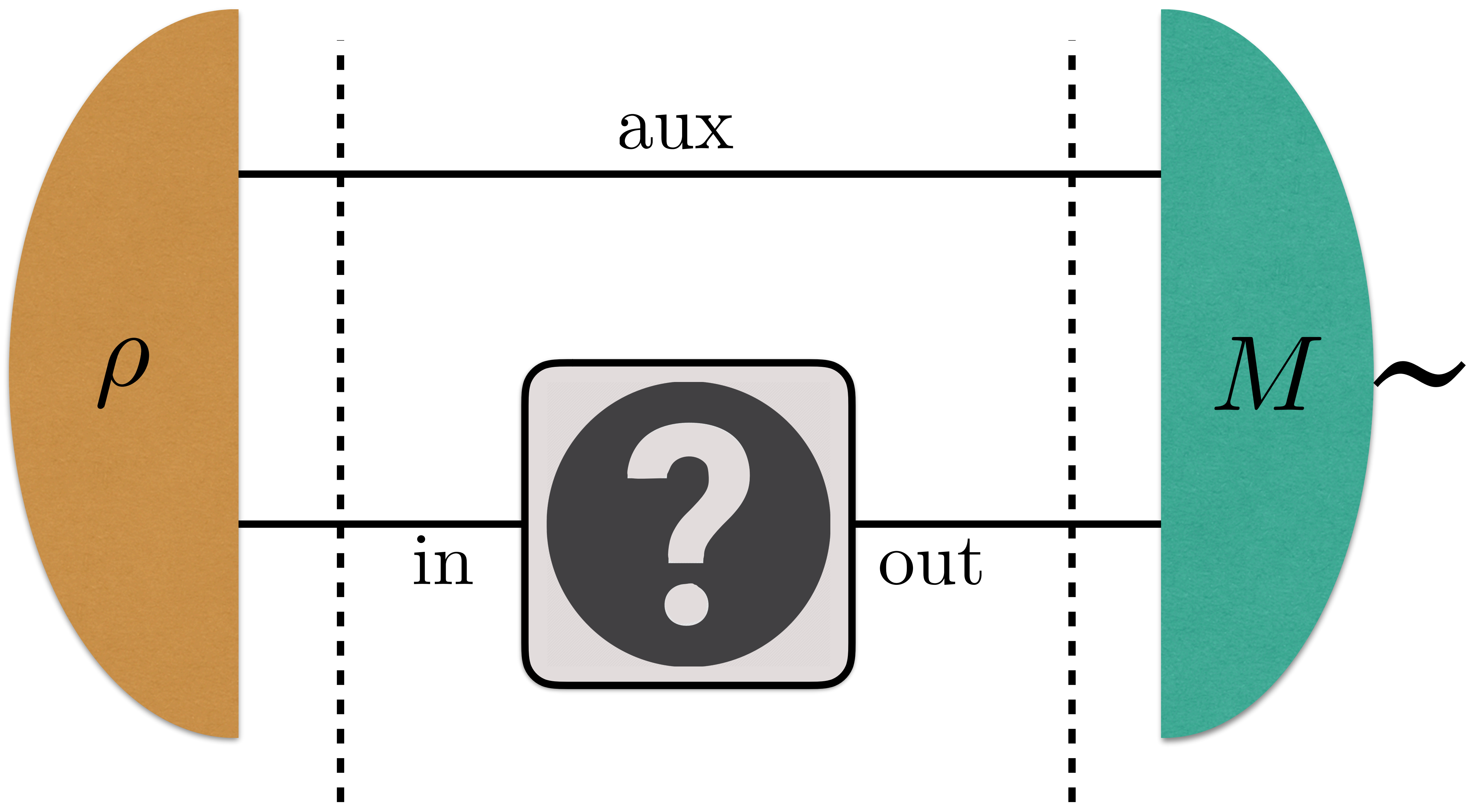

$$\rho = |\Phi^+\rangle\langle\Phi^+| \qquad \rho' = (\mathbb{I} \otimes \sigma_i)|\Phi^+\rangle\langle\Phi^+|$$

EXAMPLE

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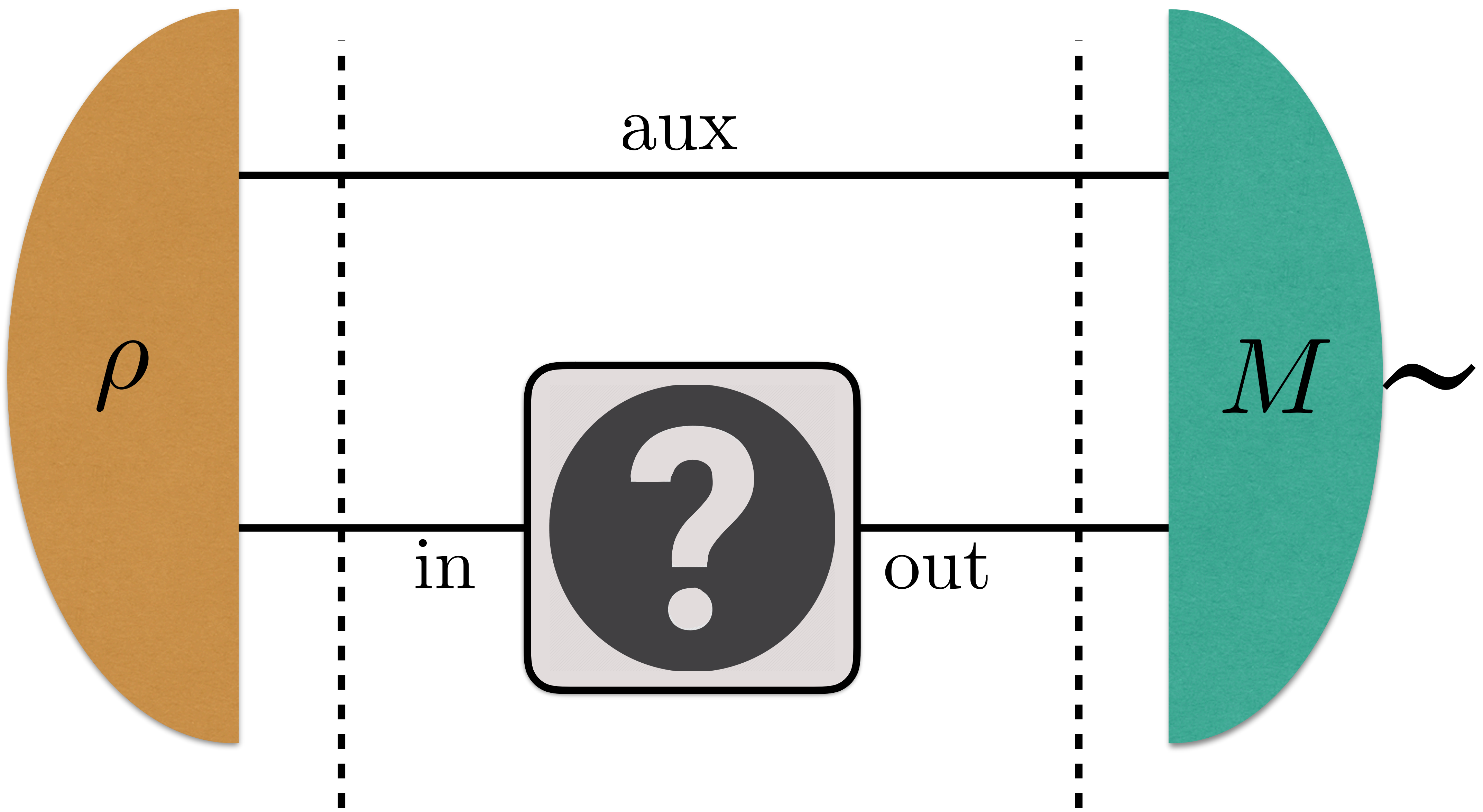

$$\rho = |\Phi^+\rangle\langle\Phi^+| \qquad \rho' = (\mathbb{I}\otimes\sigma_i)|\Phi^+\rangle\langle\Phi^+|$$

$$(\mathbb{I}\otimes\mathbb{I})|\Phi^+\rangle\langle\Phi^+| = |\Phi^+\rangle\langle\Phi^+|$$

$$(\mathbb{I}\otimes\sigma_X)|\Phi^+\rangle\langle\Phi^+| = |\Psi^+\rangle\langle\Psi^+|$$

$$(\mathbb{I}\otimes\sigma_Y)|\Phi^+\rangle\langle\Phi^+| = |\Psi^-\rangle\langle\Psi^-|$$

$$(\mathbb{I}\otimes\sigma_Z)|\Phi^+\rangle\langle\Phi^+| = |\Phi^-\rangle\langle\Phi^-|$$

EXAMPLE

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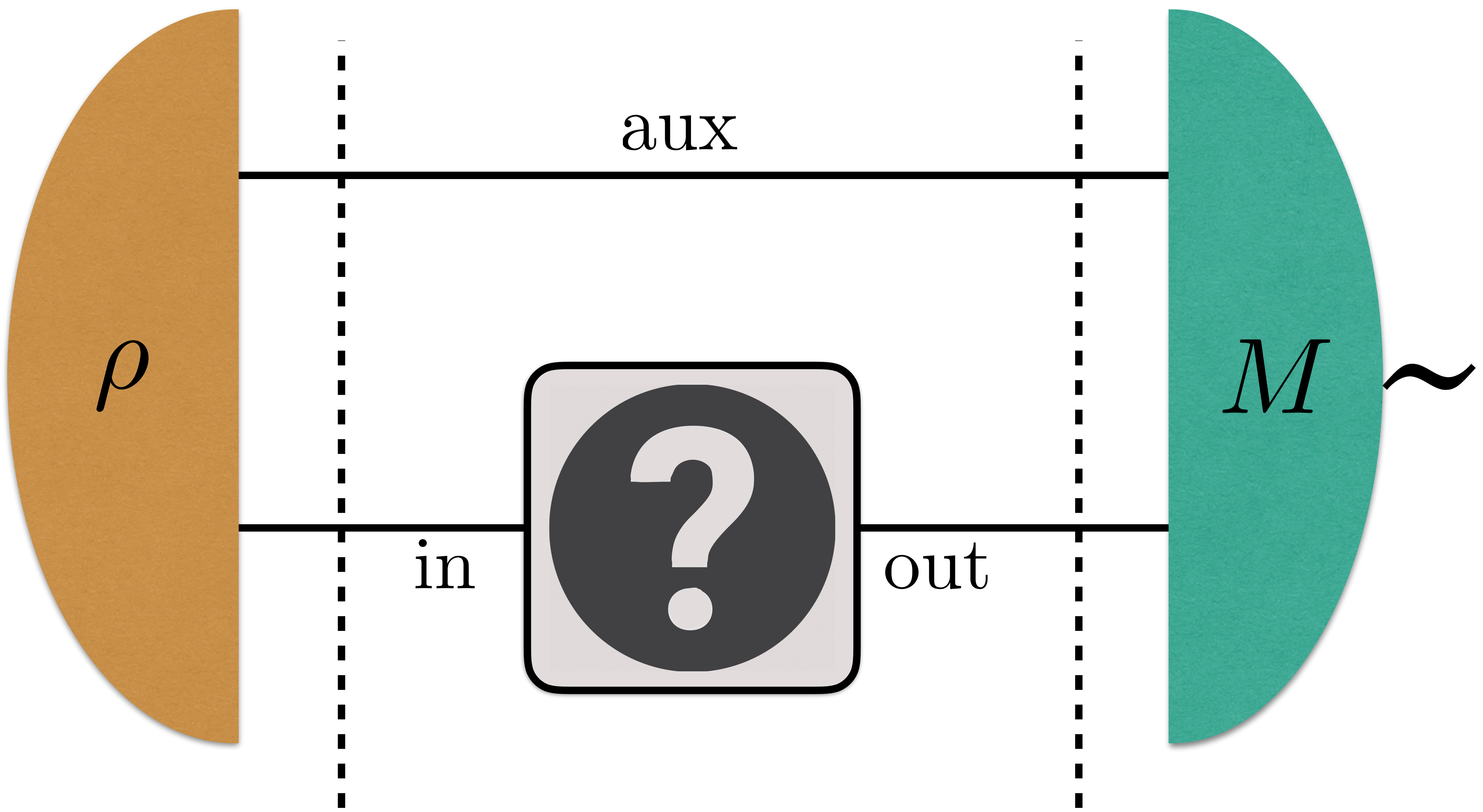

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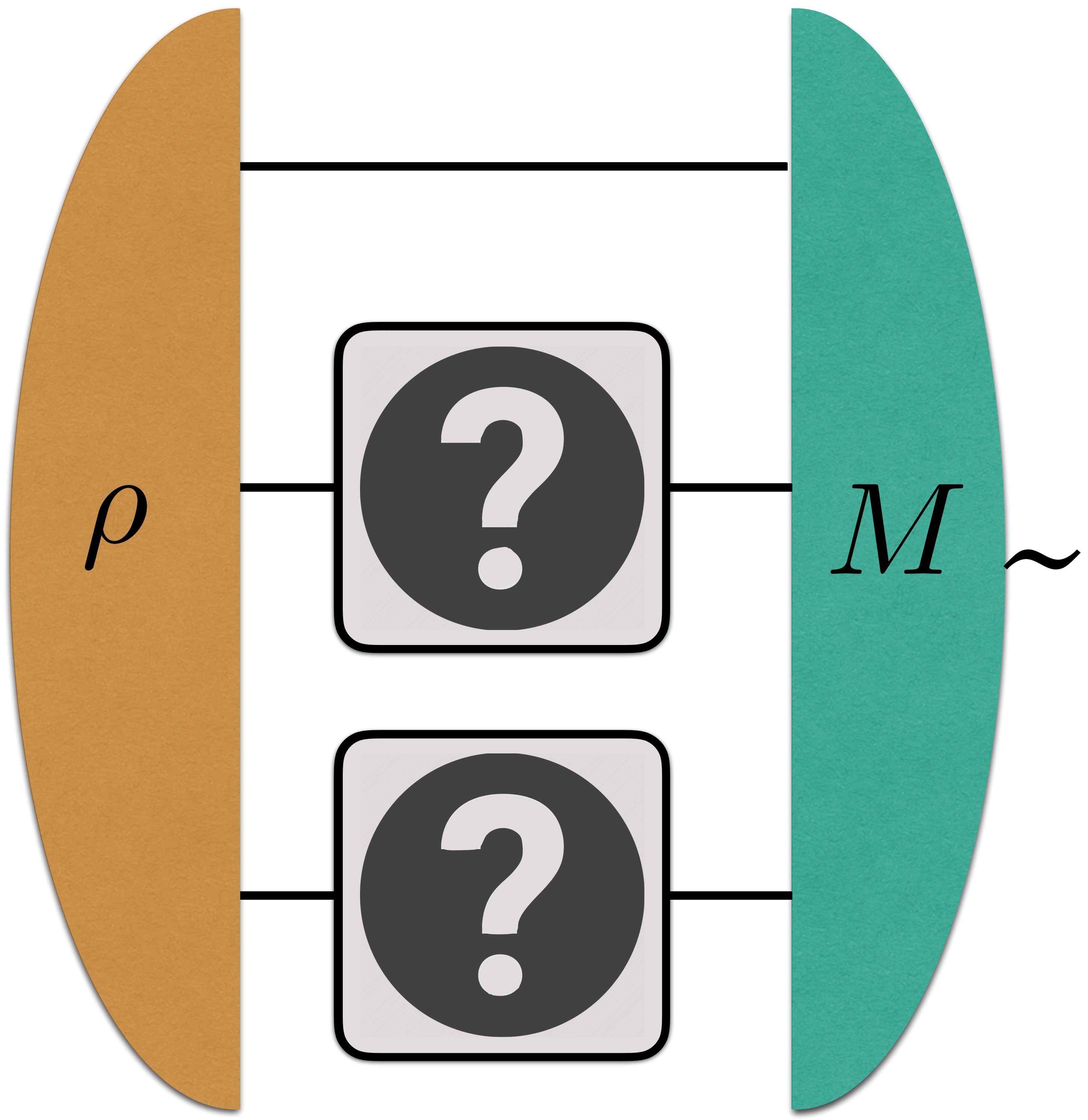
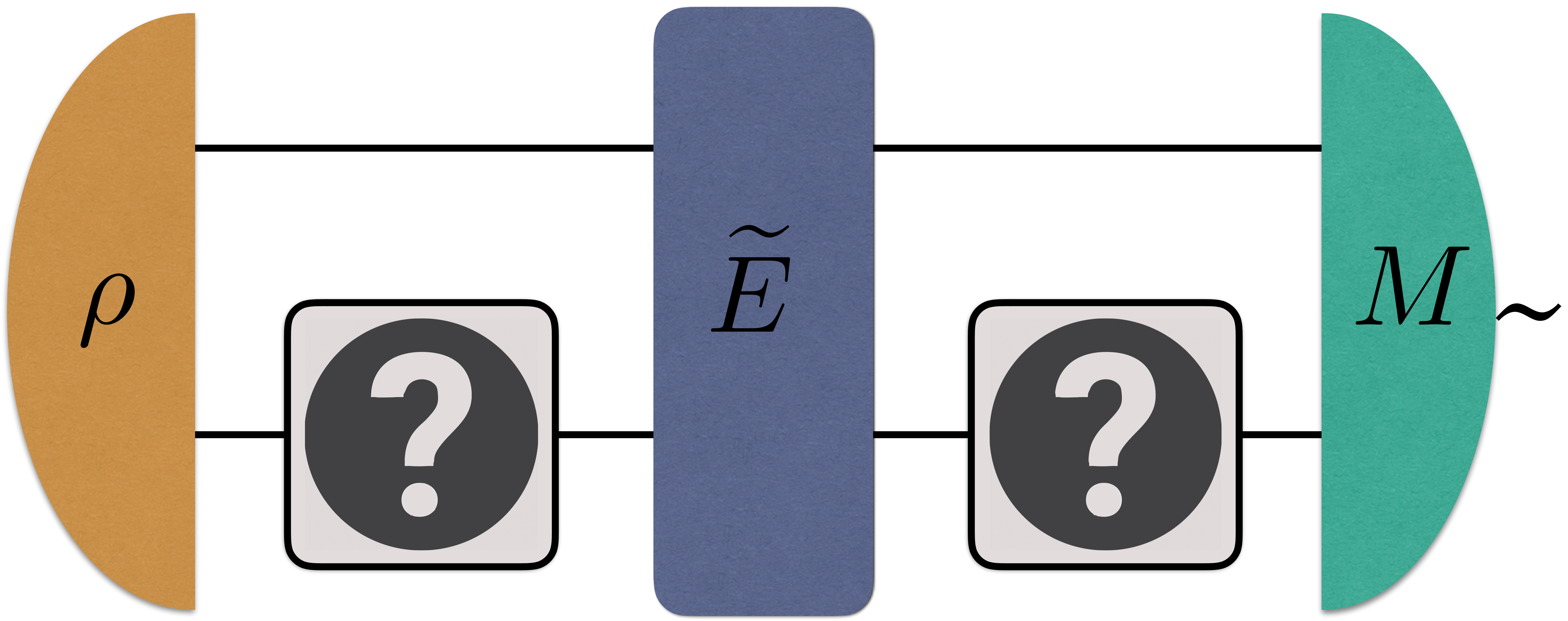
$$P = 1$$

$$(\mathbb{I} \otimes \mathbb{I})|\Phi^+\rangle\langle\Phi^+| = |\Phi^+\rangle\langle\Phi^+|$$
$$(\mathbb{I} \otimes \sigma_X)|\Phi^+\rangle\langle\Phi^+| = |\Psi^+\rangle\langle\Psi^+|$$
$$(\mathbb{I} \otimes \sigma_Y)|\Phi^+\rangle\langle\Phi^+| = |\Psi^-\rangle\langle\Psi^-|$$
$$(\mathbb{I} \otimes \sigma_Z)|\Phi^+\rangle\langle\Phi^+| = |\Phi^-\rangle\langle\Phi^-|$$


ρ
M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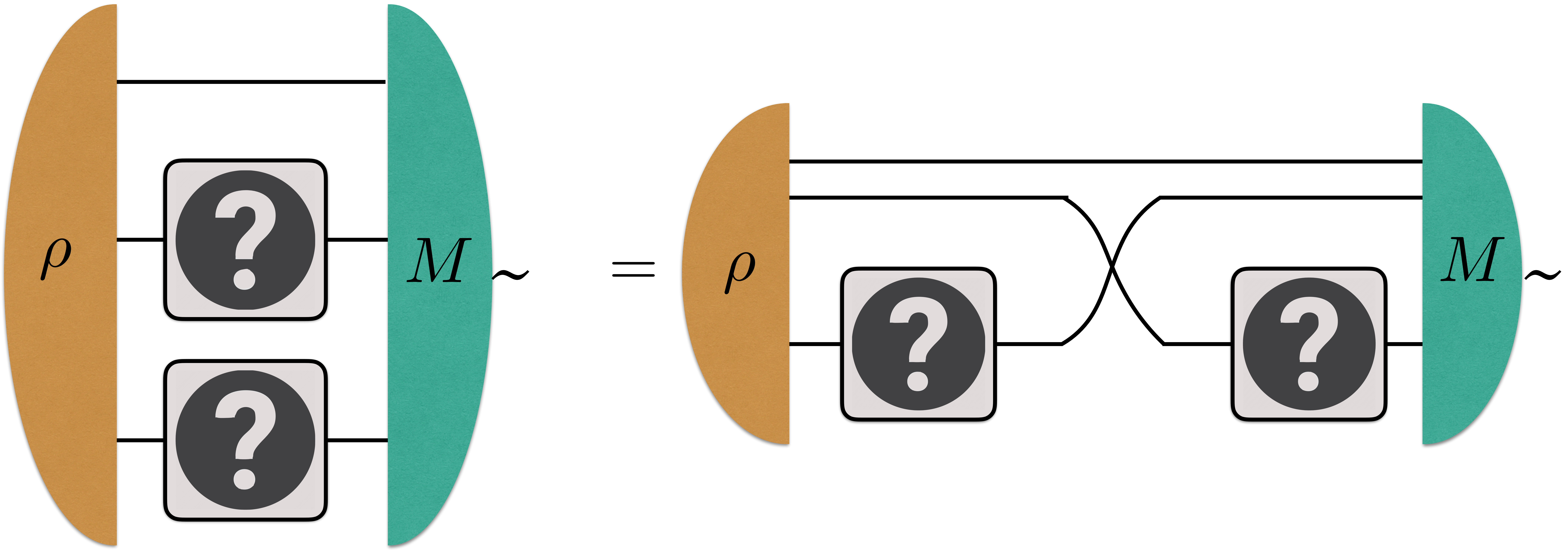
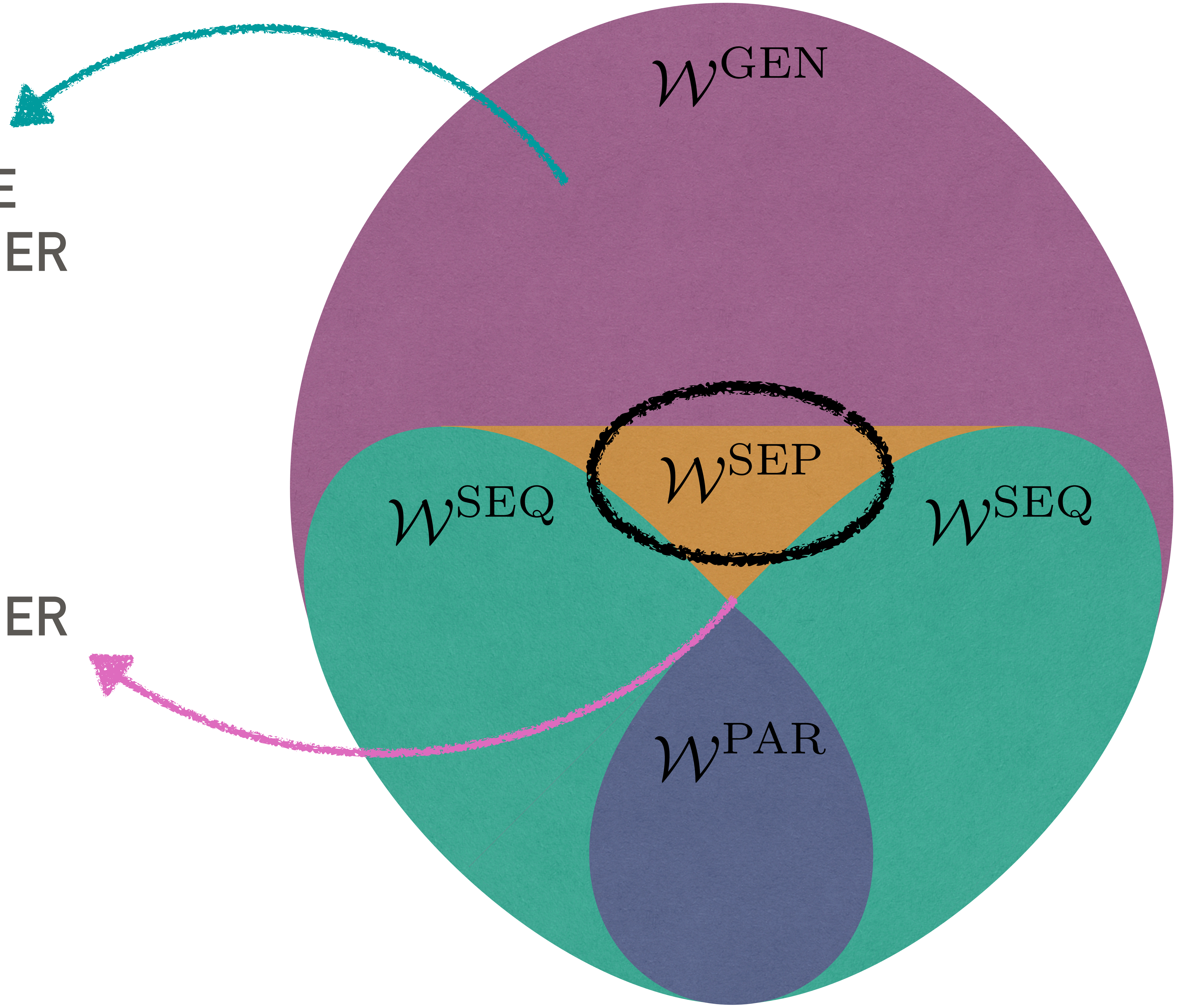

ρ
M
~
=
ρ
M
~

$$W \geq 0$$
$$\mathrm{Tr}[W(C_1 \otimes C_2)] = 1$$
$$\forall \;\; C_1, C_2$$

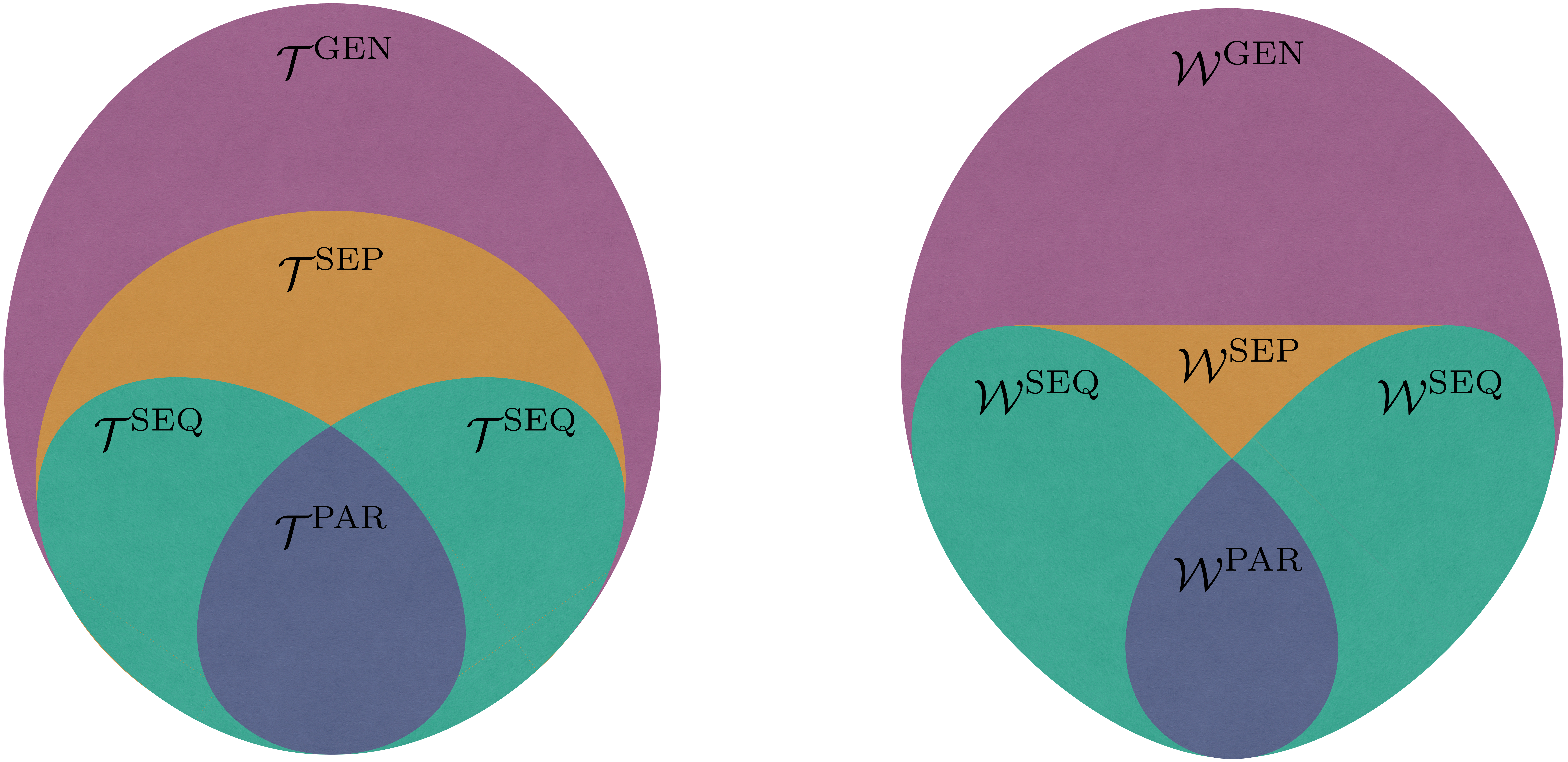
INDEFINITE CAUSAL ORDER

DEFINITE CAUSAL ORDER



Set of all processes $\mathcal{W}^{\mathcal{S}}$

O. Oreshkov, F. Costa, and Č. Brukner, Nat. Commun. 3, 1092 (2012), arXiv:1105.4464 [quant-ph].



Set of all testers $\mathcal{T}^{\mathcal{S}}$

Set of all processes $\mathcal{W}^{\mathcal{S}}$

SEMIDEFINITE PROGRAMMING (SDP)

$$P^{\mathcal{S}} = \max_{\{T_i^{\mathcal{S}}\}} \sum_i p_i \, \mathrm{Tr}\left(C_i^{\otimes 2} \, T_i^{\mathcal{S}}\right)$$

PRIMAL

given $\{p_i, C_i\}$

maximize $\sum_i p_i \mathrm{Tr}(T_i^{\mathcal{S}} \, C_i^{\otimes 2})$

subject to $T_i^{\mathcal{S}} \geq 0 \; \forall \, i, \quad \sum_i T_i^{\mathcal{S}} = W^{\mathcal{S}}$

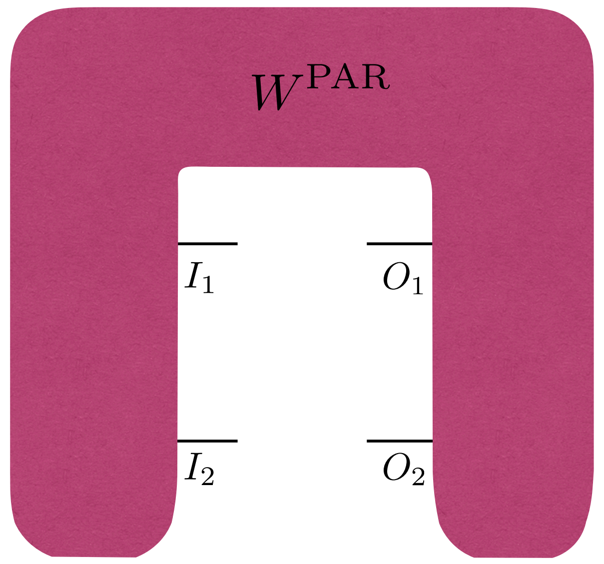
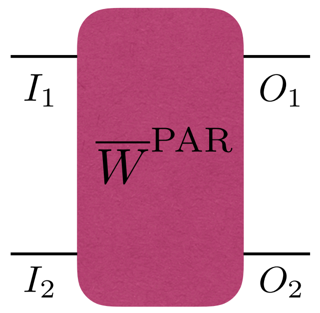
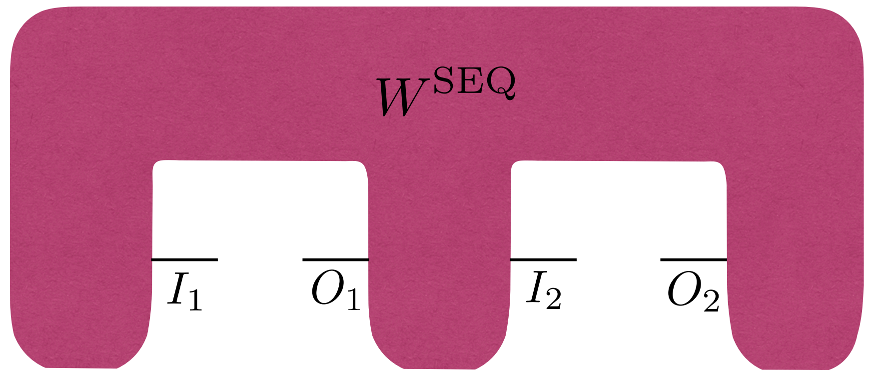
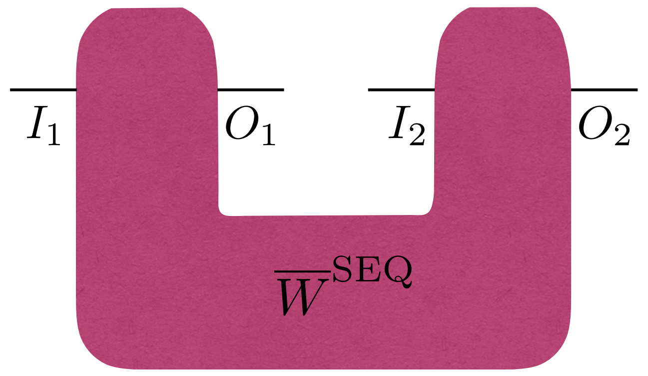
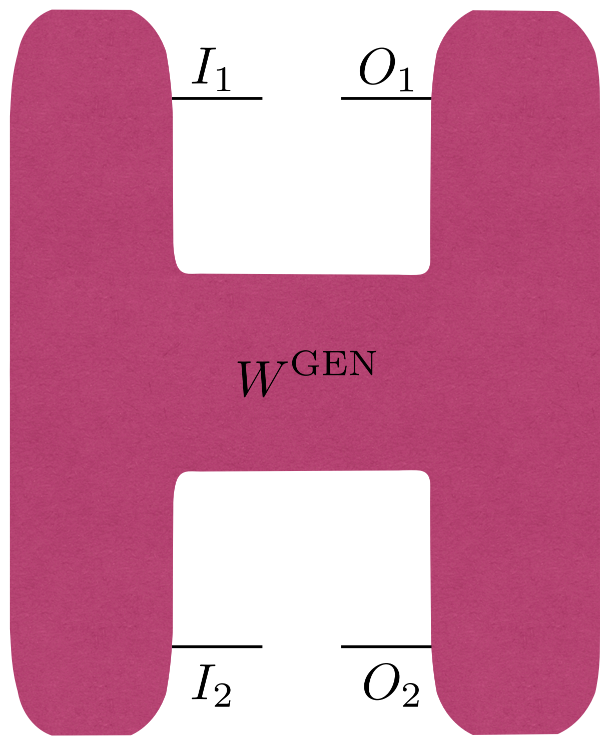
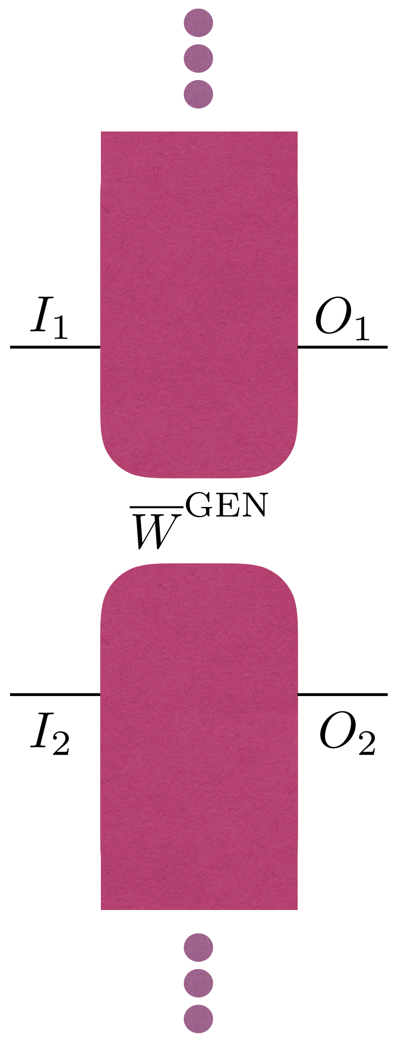
DUAL

given $\{p_i, C_i\}$

minimize λ

subject to $p_i \, C_i^{\otimes 2} \leq \lambda \, \overline{W}^{\mathcal{S}} \quad \forall \, i$

$$\mathrm{Tr}(W\,\overline{W}) = 1 \quad \forall \;\; W \in \mathcal{W},\; \overline{W} \in \overline{\mathcal{W}}$$

	PROCESS		DUAL AFFINE (CHANNEL)	
PARALLEL		$\mathrm{Tr}(W^{\text{PAR}}) = d_{O_1} d_{O_2}$ $W^{\text{PAR}} = {}_{O_1O_2}W^{\text{PAR}}$		$\mathrm{Tr}(\overline{W}^{\text{PAR}}) = d_{I_1} d_{I_2}$ ${}_{O_1O_2}\overline{W}^{\text{PAR}} = {}_{I_1O_1I_2O_2}\overline{W}^{\text{PAR}}$
SEQUENTIAL		$\mathrm{Tr}(W^{\text{SEQ}}) = d_{O_1} d_{O_2}$ $W^{\text{SEQ}} = {}_{O_2}W^{\text{SEQ}}$ ${}_{I_2O_2}W^{\text{SEQ}} = {}_{O_1I_2O_2}W^{\text{SEQ}}$		$\mathrm{Tr}(\overline{W}^{\text{SEQ}}) = d_{I_1} d_{I_2}$ ${}_{O_2}\overline{W}^{\text{SEQ}} = {}_{I_2O_2}\overline{W}^{\text{SEQ}}$ ${}_{O_1I_2O_2}\overline{W}^{\text{SEQ}} = {}_{I_1O_1I_2O_2}\overline{W}^{\text{SEQ}}$
GENERAL		$\mathrm{Tr}(W^{\text{GEN}}) = d_{O_1} d_{O_2}$ ${}_{I_1O_1}W^{\text{GEN}} = {}_{I_1O_1O_2}W^{\text{GEN}}$ ${}_{I_2O_2}W^{\text{GEN}} = {}_{O_1I_2O_2}W^{\text{GEN}}$ $W^{\text{GEN}} = {}_{O_1}W^{\text{GEN}} + {}_{O_2}W^{\text{GEN}} - {}_{O_1O_2}W^{\text{GEN}}$		$\mathrm{Tr}(\overline{W}^{\text{GEN}}) = d_{I_1} d_{I_2}$ ${}_{O_1}\overline{W}^{\text{GEN}} = {}_{I_1O_1}\overline{W}^{\text{GEN}}$ ${}_{O_2}\overline{W}^{\text{GEN}} = {}_{I_2O_2}\overline{W}^{\text{GEN}}$

COMPUTER-ASSISTED PROOF

Example: how to create a “valid” channel

- Take numerically imprecise matrix C from the solution of an SDP
- Truncate C and define $C \mapsto \frac{C + C^\dagger}{2}$
- Project C onto the subspace of valid channels, $C \mapsto L(C)$
- Find coefficient η such that $C \mapsto \eta C + (1-\eta)\mathbb{I} \geq 0$
- Output $C \mapsto d_I \frac{C}{\mathrm{Tr}(C)}$