Designing optimal protocols in Bayesian parameter estimation with higher-order operations

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PRResearch 6, 023305 (2024), arXiv:2311.01513 [quant-ph]

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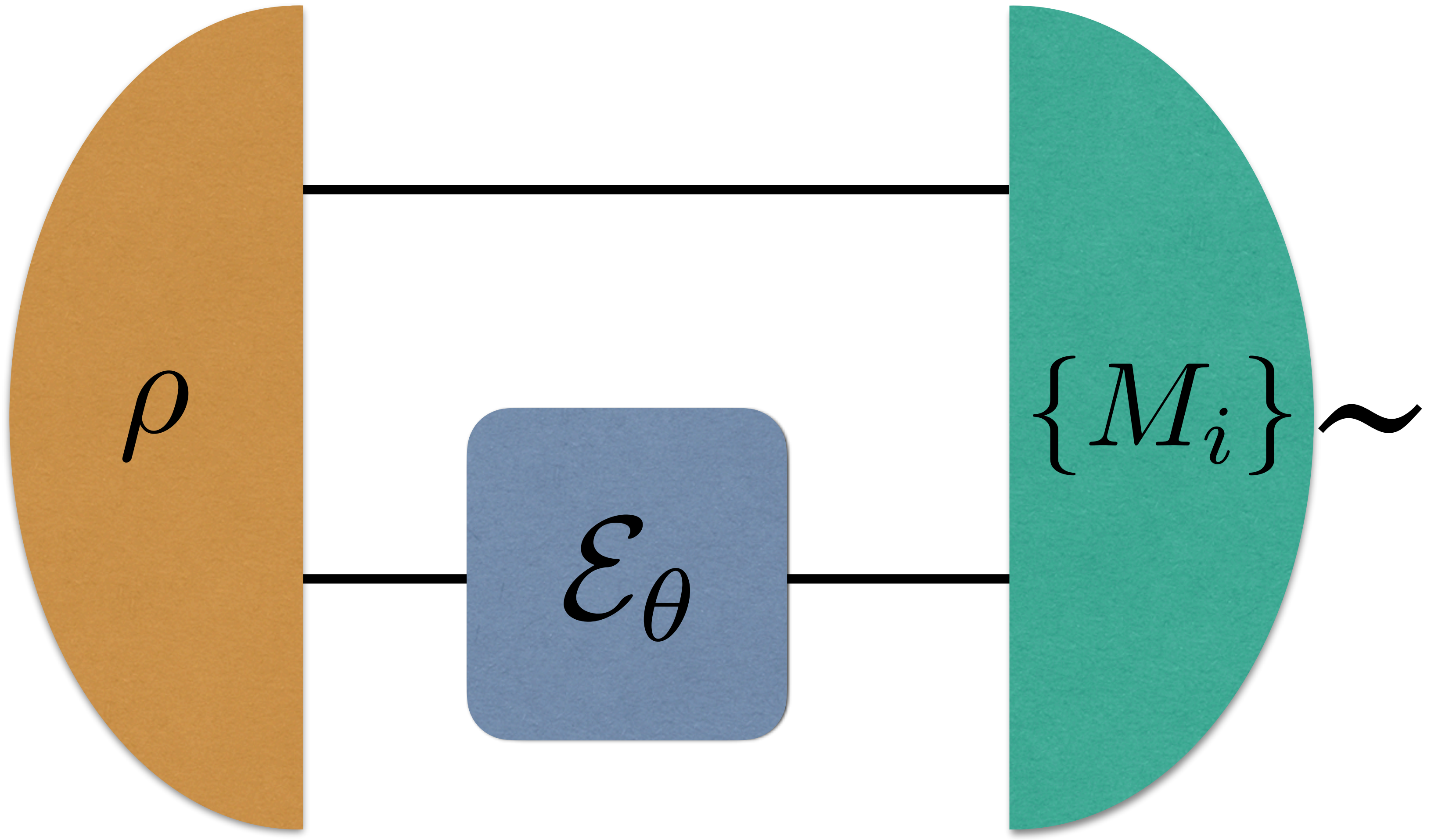
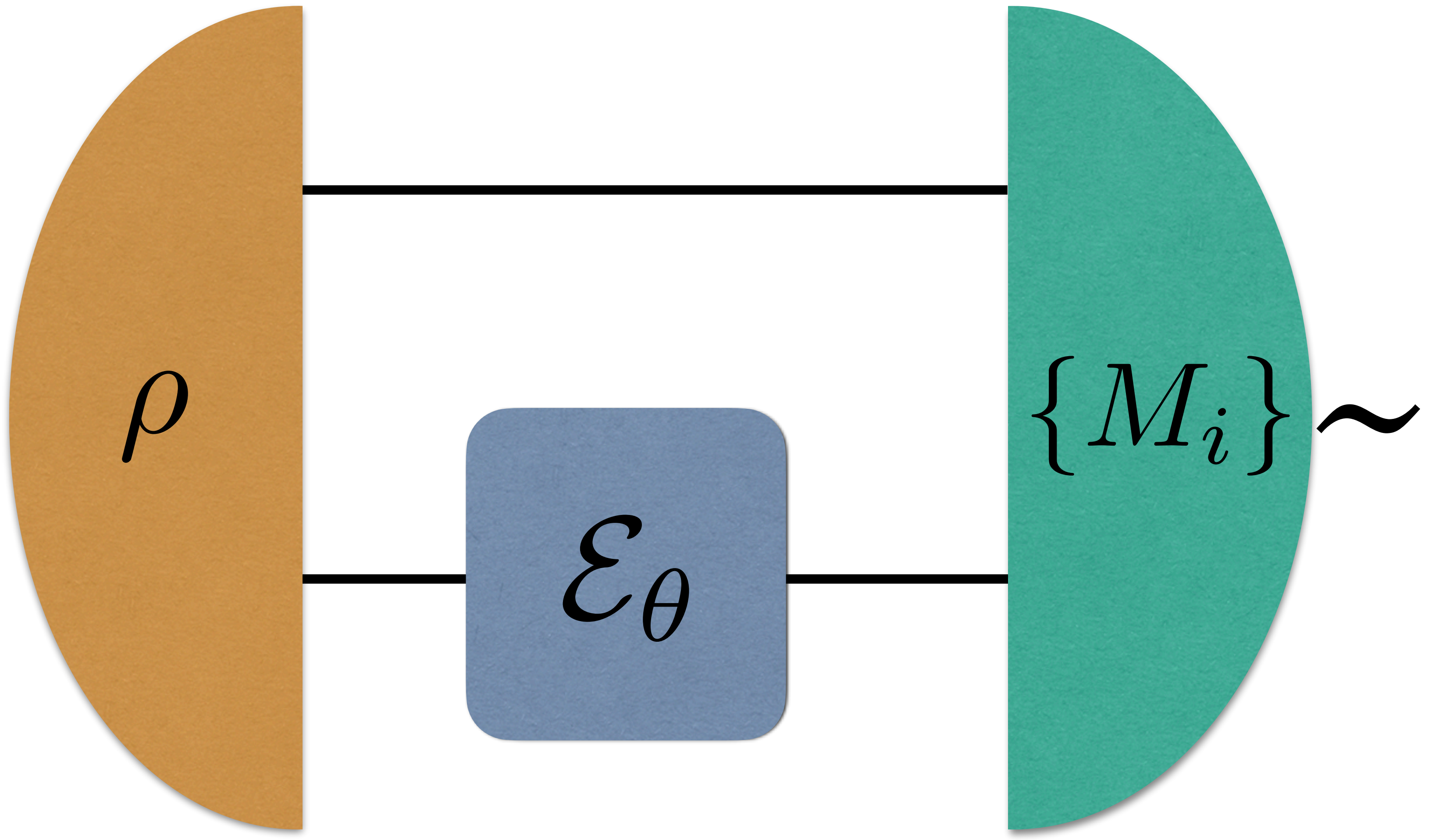
the task:

single-shot Bayesian quantum parameter estimation

Single-shot Bayesian quantum parameter estimation



Single-shot Bayesian quantum parameter estimation



given

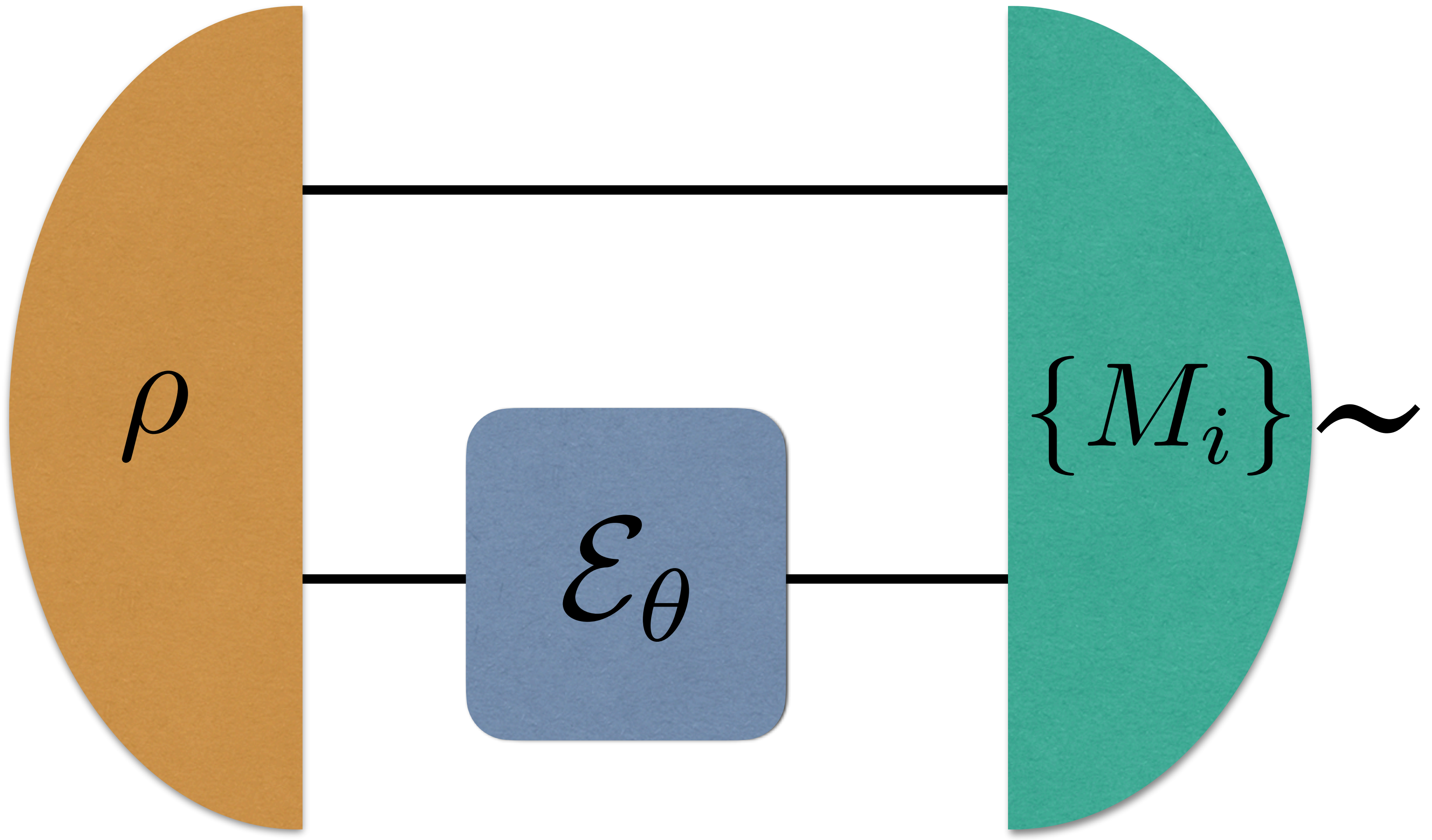
parameter: $\theta \in [\theta_{\min}, \theta_{\max}]$

channel: $\{\mathcal{E}_\theta\}_\theta$

prior: $p(\theta)$

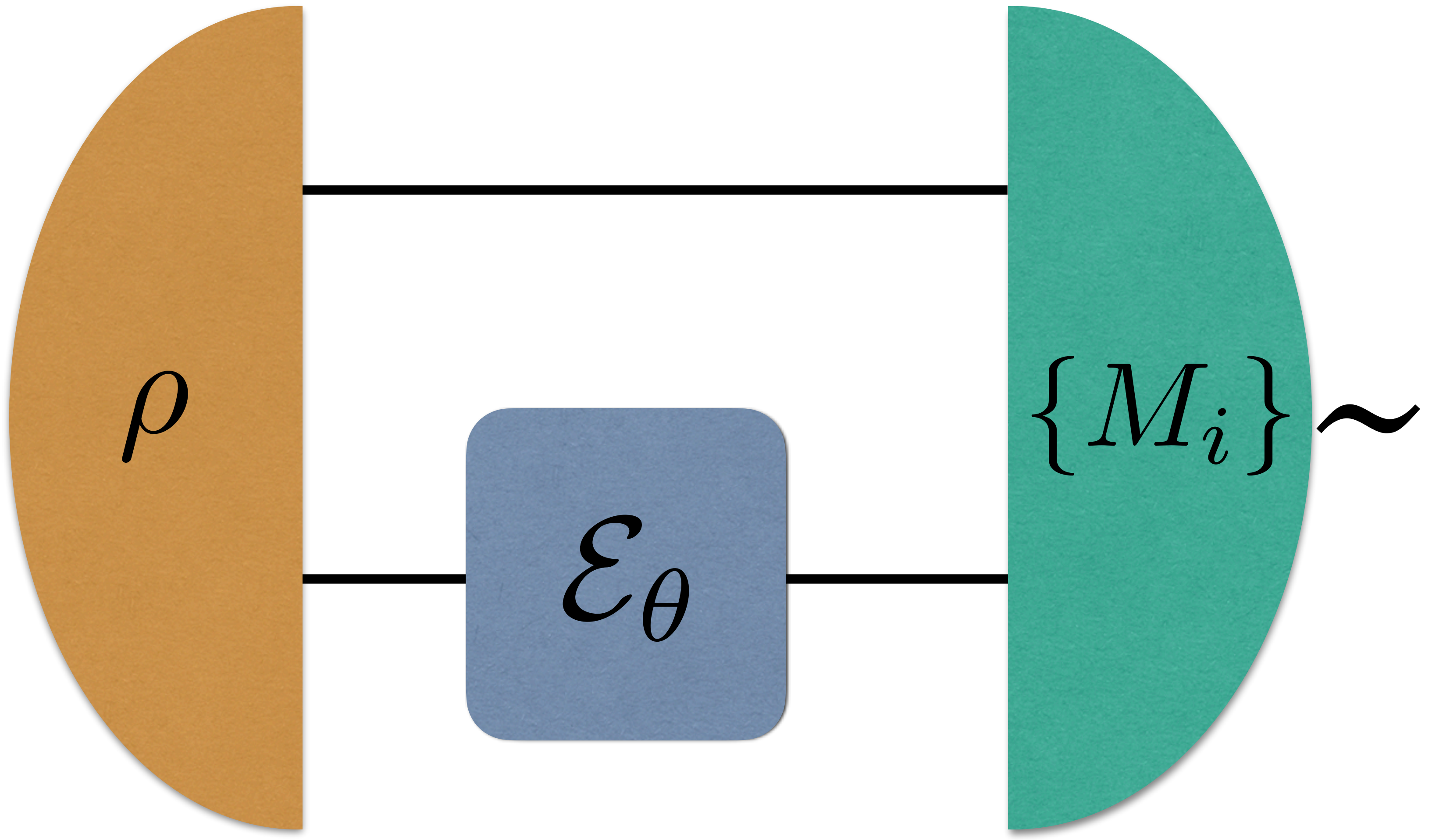
cost/reward function: $r(\theta, \hat{\theta}_i)$

Single-shot Bayesian quantum parameter estimation



given		variables	
parameter:	$\theta \in [\theta_{\min}, \theta_{\max}]$	probe state:	ρ
channel:	$\{\mathcal{E}_\theta\}_\theta$	measurement:	$\{M_i\}_{i=1}^{N_O}$
prior:	$p(\theta)$	estimator:	$f(i) = \hat{\theta}_i$
cost/reward function:	$r(\theta, \hat{\theta}_i)$		

Single-shot Bayesian quantum parameter estimation



$$\mathcal{S} = \sum_{i=1}^{N_O} \int d\theta \, p(\theta) \, r(\theta, \hat{\theta}_i) \, \mathrm{tr}\left((\mathcal{E}_\theta \otimes \mathrm{id})[\rho] \, M_i \right)$$

given		**variables**	
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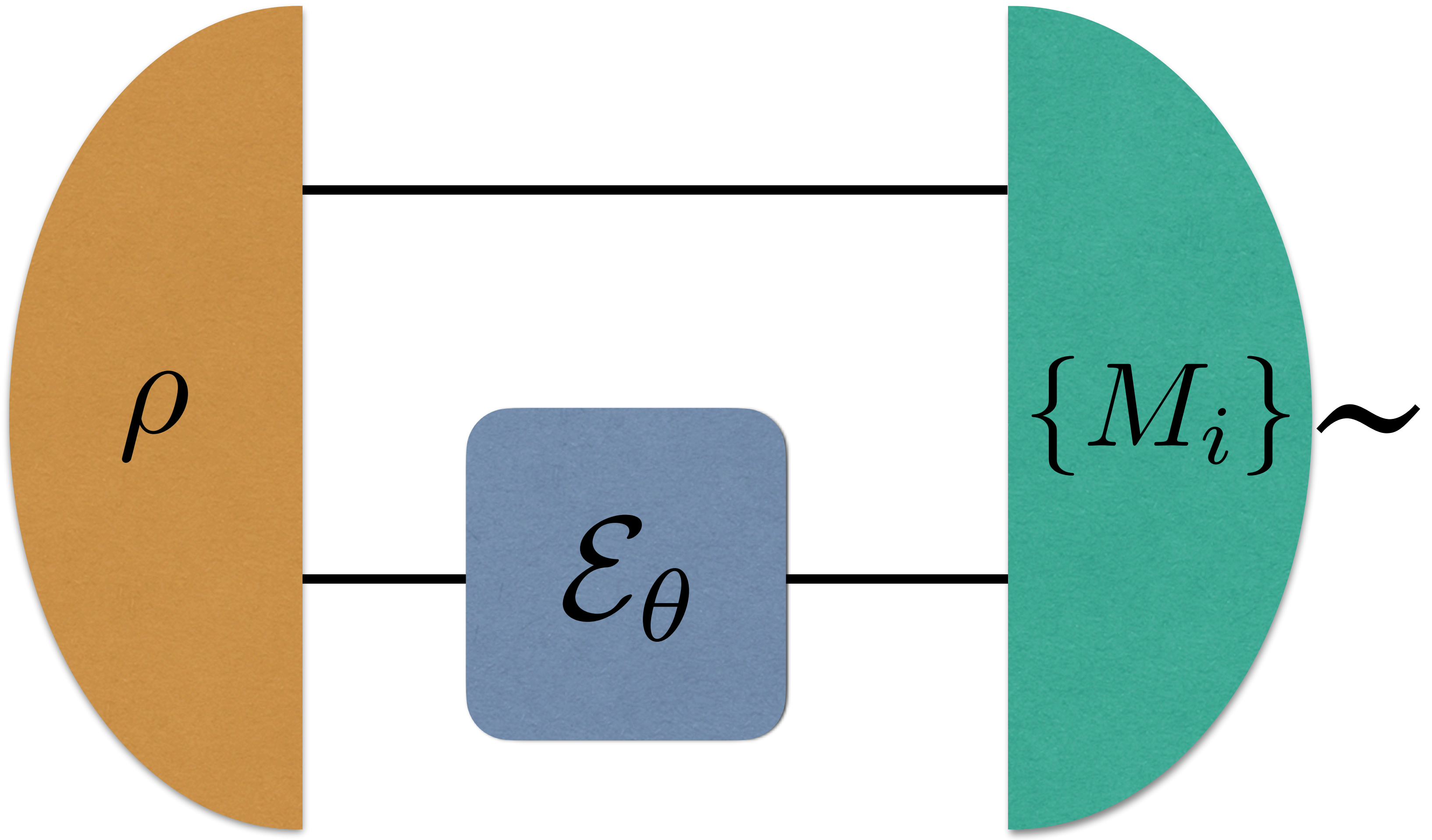

$$\{\rho, \{M_i\}, \{\hat\theta_i\}\}$$

triplet of state, measurement and estimator

$$\max_{\rho,\{M_i\},\{\hat\theta_i\}} \mathcal{S} = \max_{\rho,\{M_i\},\{\hat\theta_i\}} \sum_{i=1}^{N_O} \int d\theta \, p(\theta) \, r(\theta, \hat\theta_i) \, \mathrm{tr}\left((\mathcal{E}_\theta \otimes \mathrm{id})[\rho] \, M_i\right)$$

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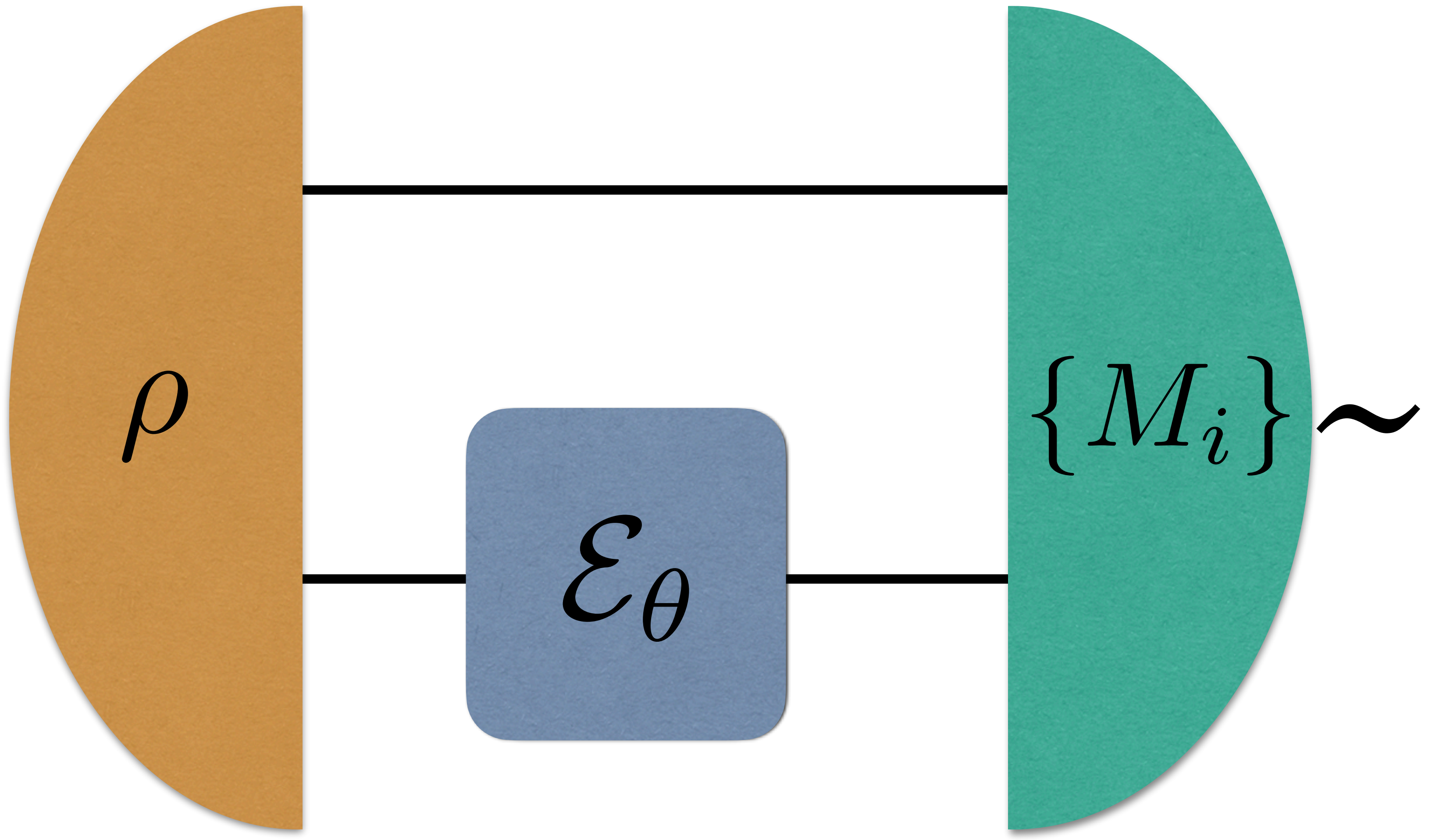

$$\{\rho, \{M_i\}, \{\hat{\theta}_i\}\}$$

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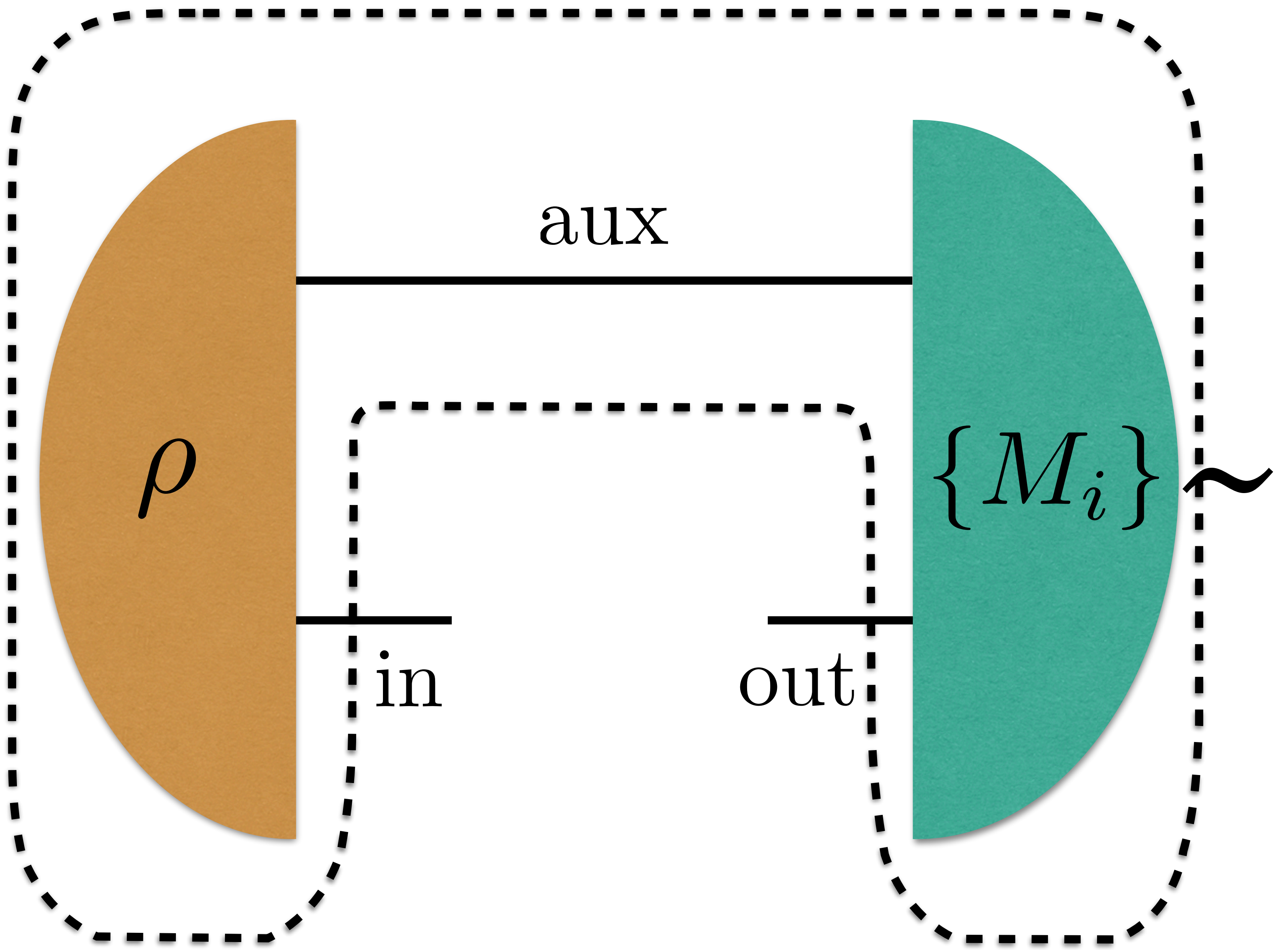
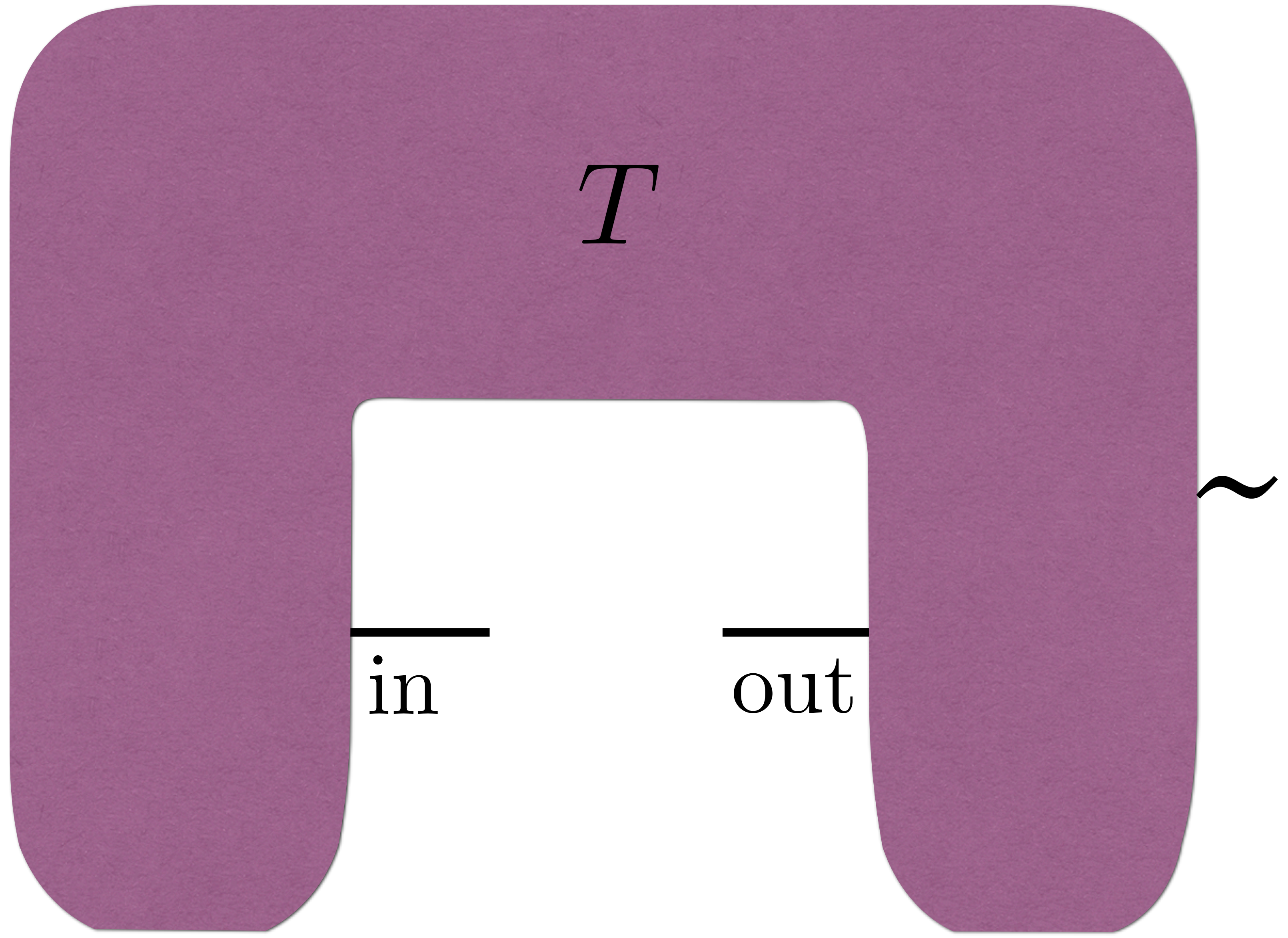
Generally a very hard problem :(

$$\max_{\rho,\{M_i\},\{\hat{\theta}_i\}} \mathcal{S} = \max_{\rho,\{M_i\},\{\hat{\theta}_i\}} \sum_{i=1}^{N_O} \int d\theta \, p(\theta) \, r(\theta, \hat{\theta}_i) \, \mathrm{tr}\left((\mathcal{E}_\theta \otimes \mathrm{id})[\rho] \, M_i\right)$$

higher-order operations

Higher-order operations: quantum testers





Chiribella, D'Ariano, and Perinotti, PRL 101, 060401 (2008)
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Ziman, PRA 77, 062112 (2008)

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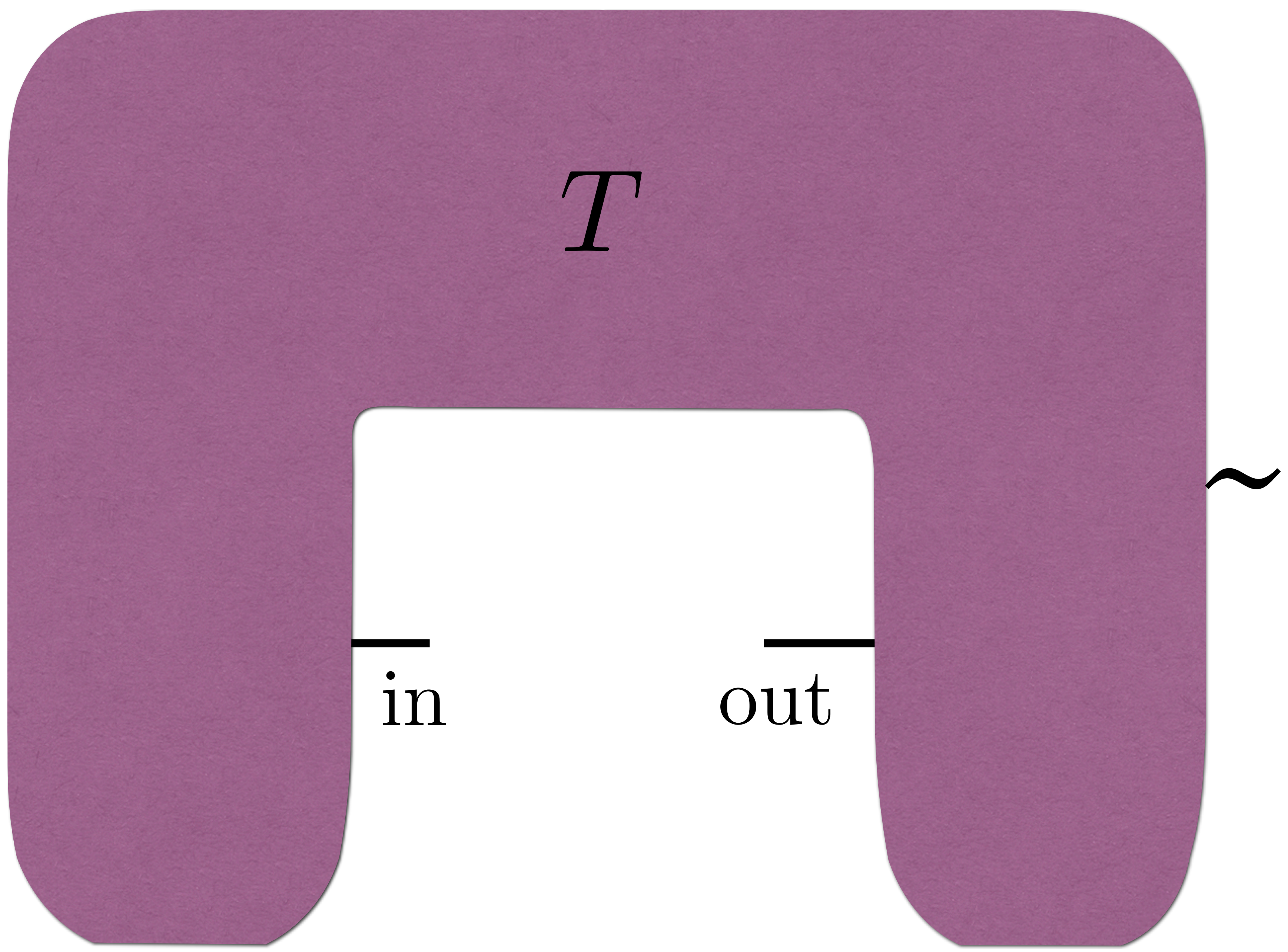

the most general set of linear maps $\widetilde{T}_i$

such that $\widetilde{T}_i : C \mapsto p(i) \quad \forall\, i$

$$=$$

the most general set of linear operators T_i

such that $p(i) = \mathrm{tr}(C\,T_i) \quad \forall\, i$

$$\Longleftrightarrow$$

$$T = \{T_i\} \;:\; T_i \in \mathcal{L}(\mathcal{H}_{\mathrm{in}} \otimes \mathcal{H}_{\mathrm{out}})$$

$$\boxed{\begin{array}{c} T_i \geq 0 \\ \sum_i T_i = \sigma^{\mathrm{in}} \otimes I^{\mathrm{out}} \end{array}}$$

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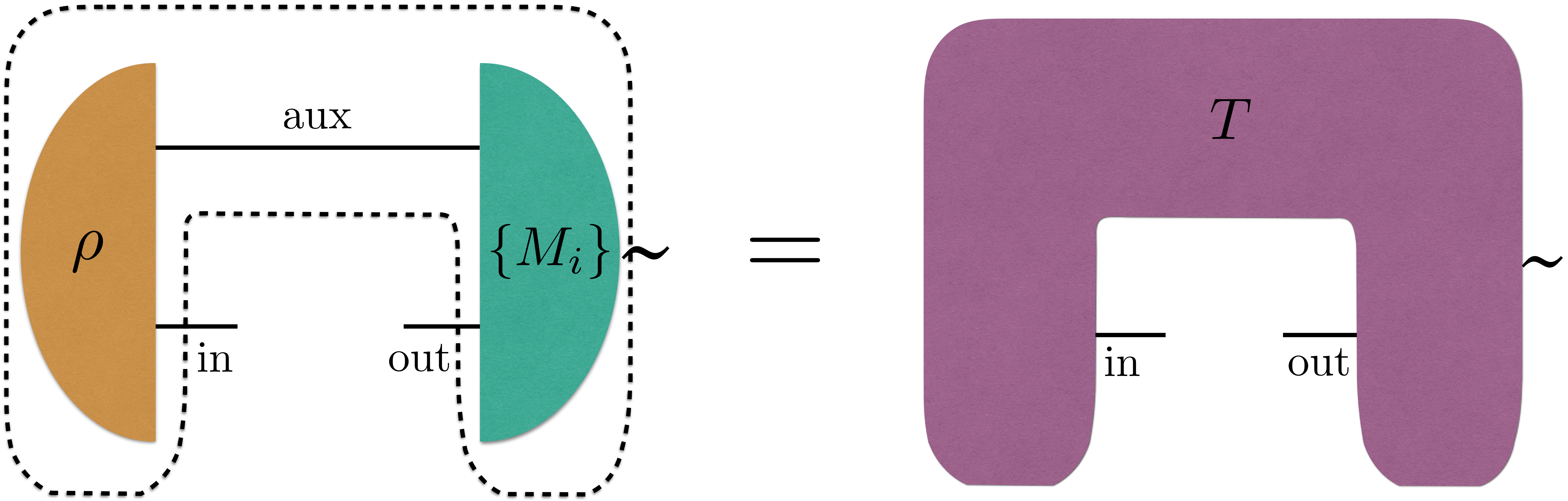
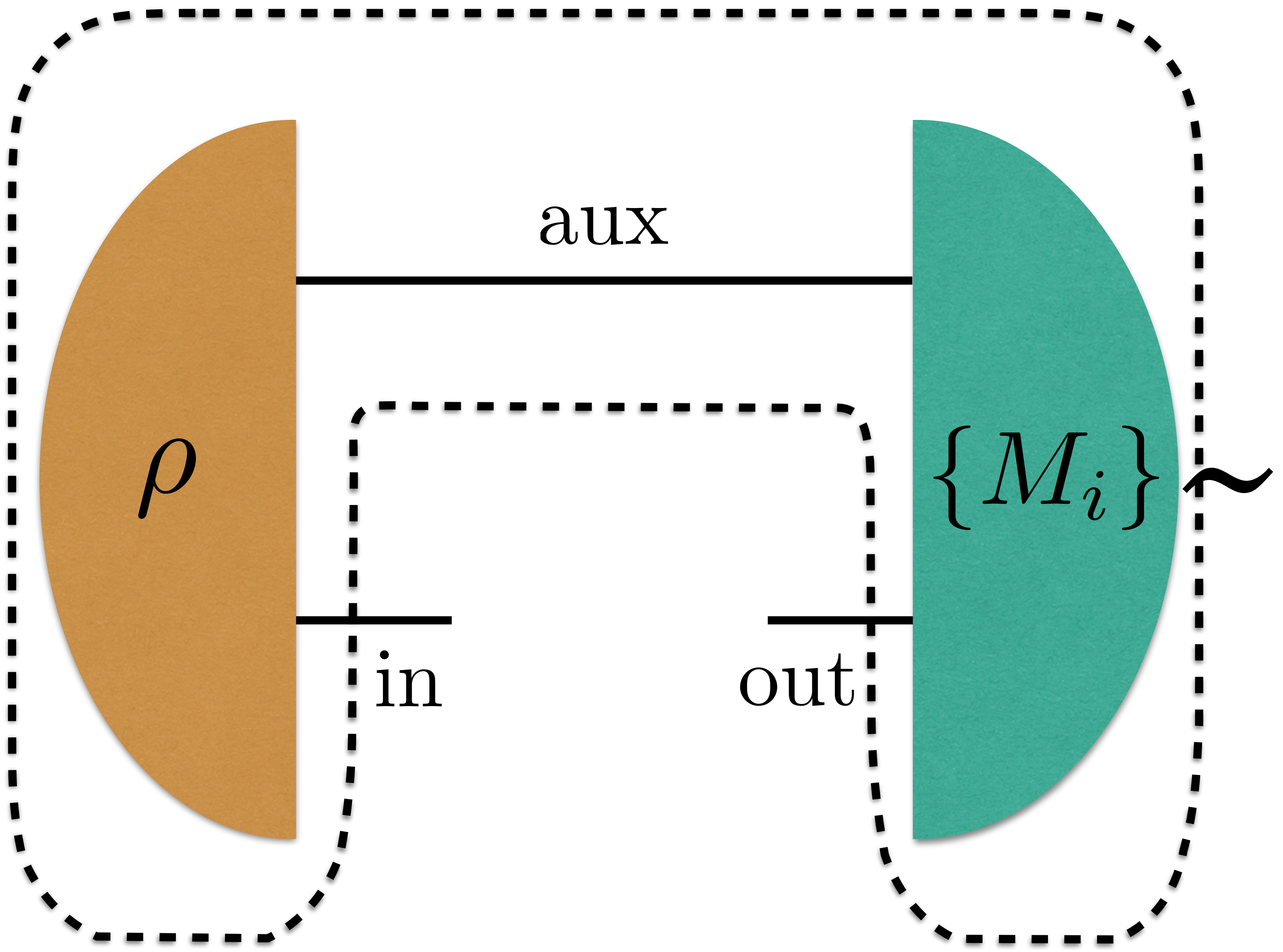


$$T_i = \mathrm{Tr}_{\mathrm{aux}}\left[(\rho^{\mathrm{in,aux}} \otimes \mathbb{I}^{\mathrm{out}})(\mathbb{I}^{\mathrm{in}} \otimes (M_i^{\mathrm{aux,out}})^{T_{\mathrm{out}}})\right]$$

$$\Longleftrightarrow$$

$$T_i \geq 0, \quad \sum_i T_i = \sigma^{\mathrm{in}} \otimes \mathbb{I}^{\mathrm{out}}$$

Higher-order operations: quantum testers



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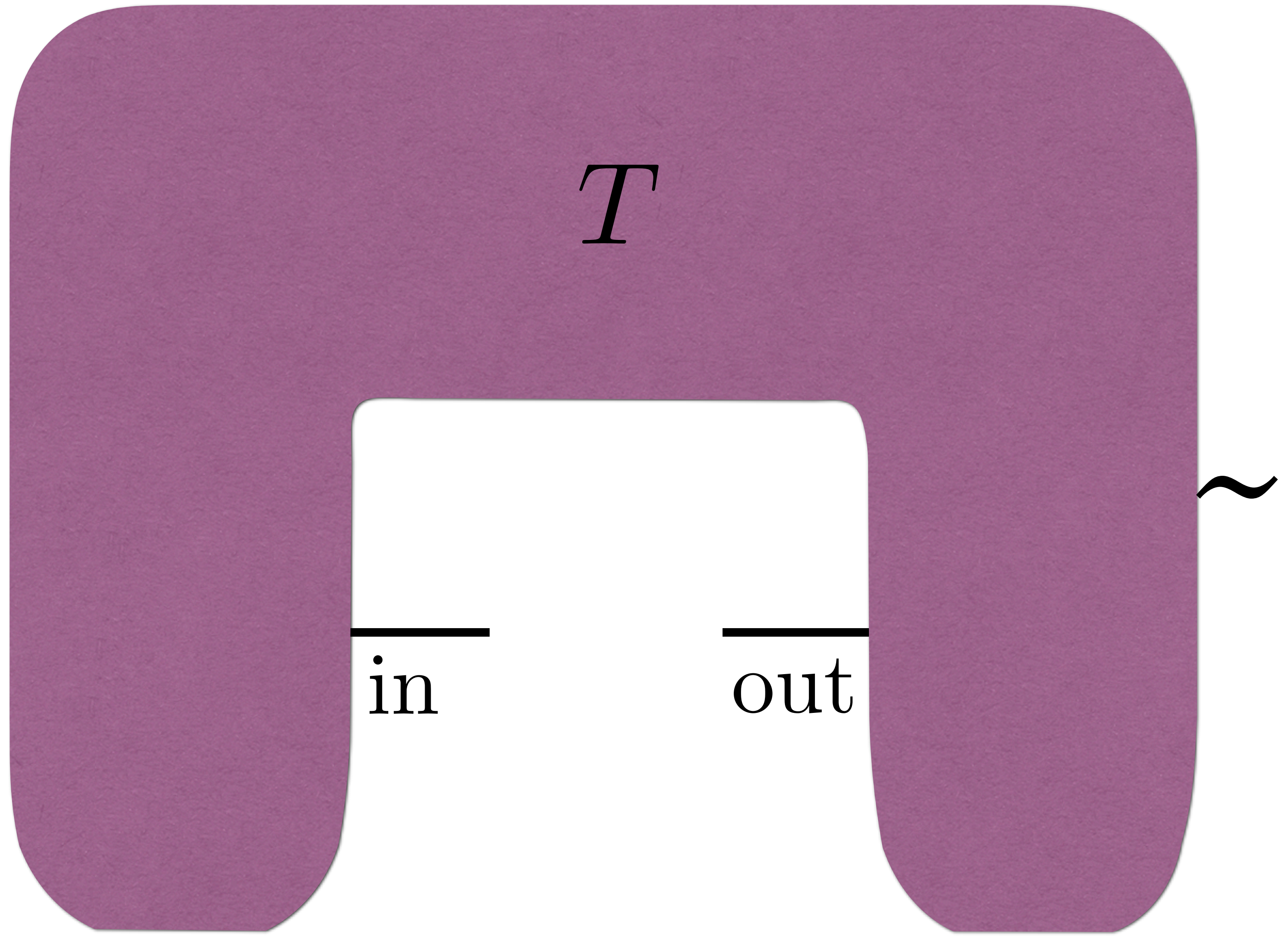

$$\rho := (\mathbb{1}^I \otimes \sqrt{\sigma}) \sum_{ij} |ii\rangle\langle jj| \, (\mathbb{1}^I \otimes \sqrt{\sigma})^\dagger$$

$$M_i := \left(\sqrt{\sigma}^{-1} \otimes \mathbb{1}^O\right) T_i \left(\sqrt{\sigma}^{-1} \otimes \mathbb{1}^O\right)^\dagger$$

$\longleftarrow$

$$T_i \geq 0$$

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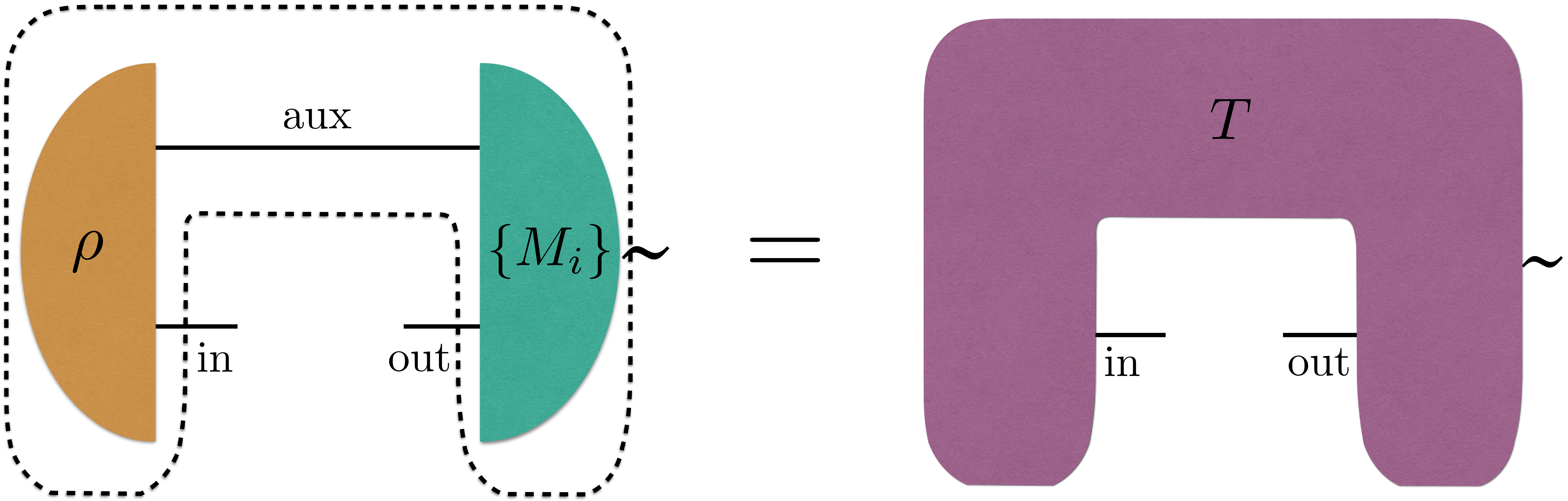

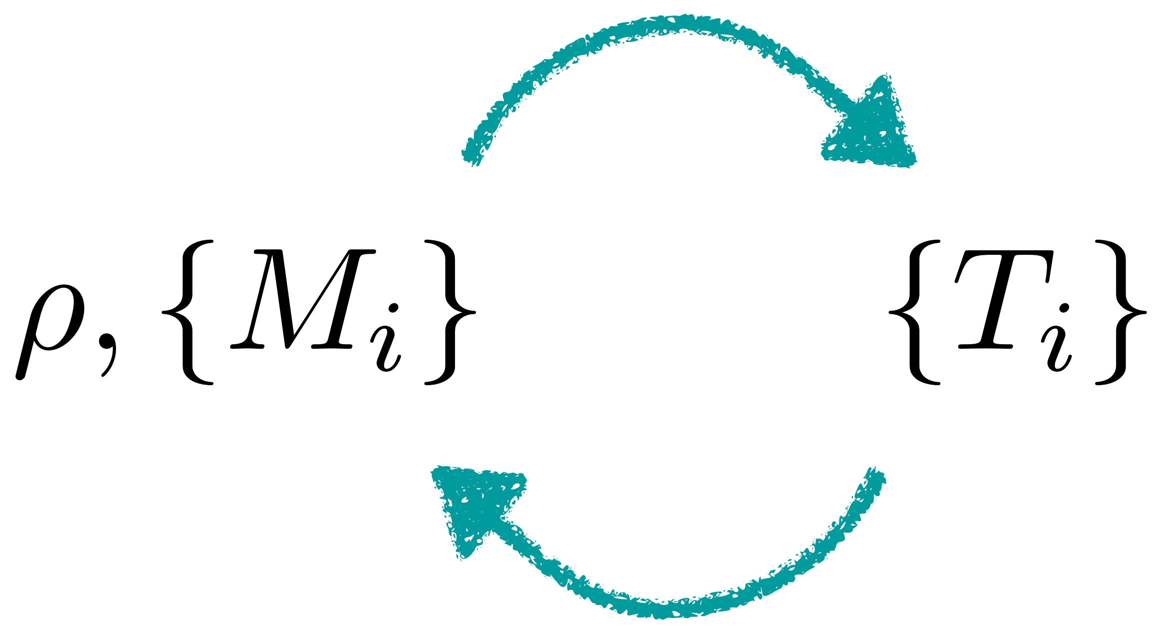

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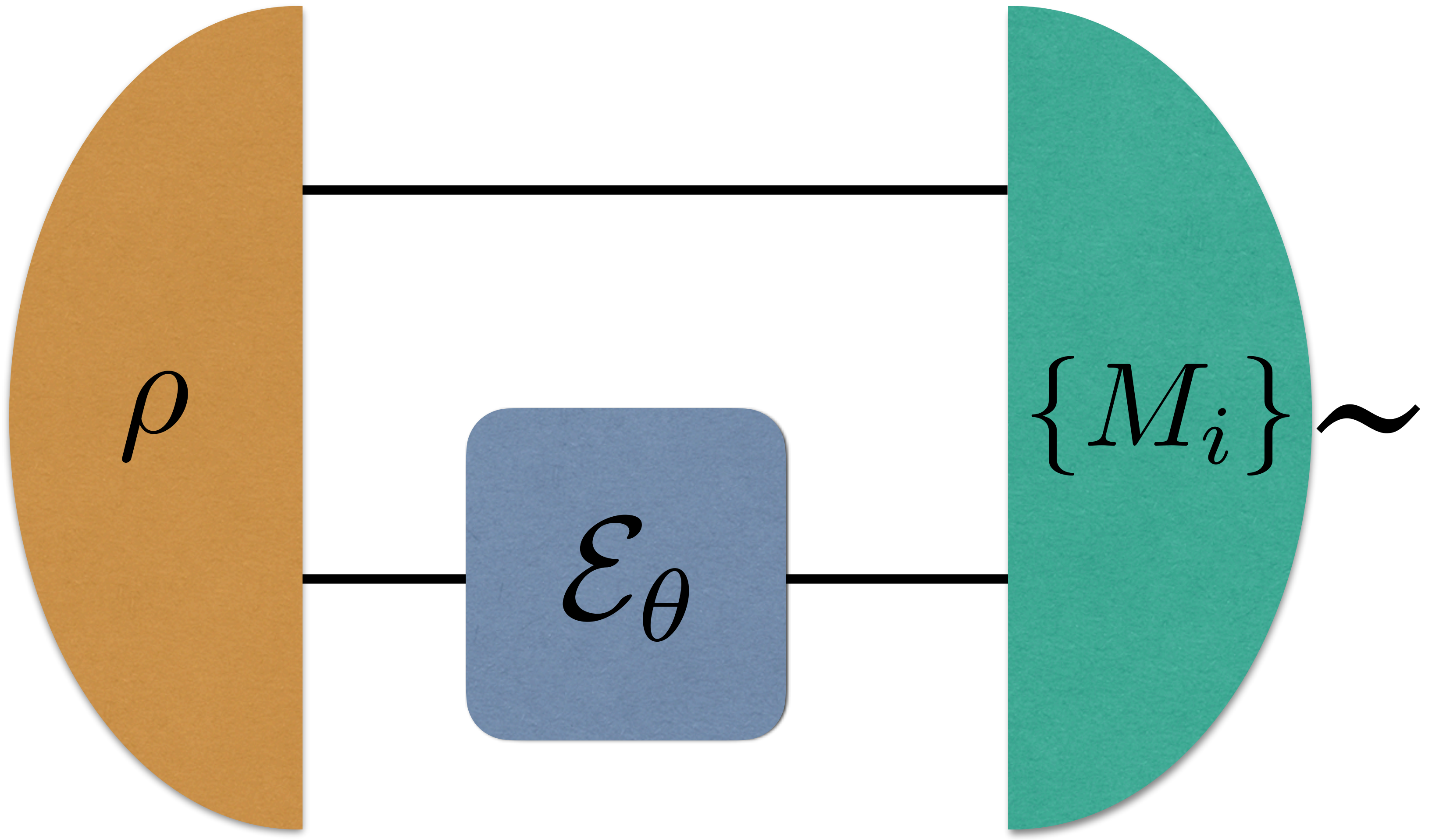

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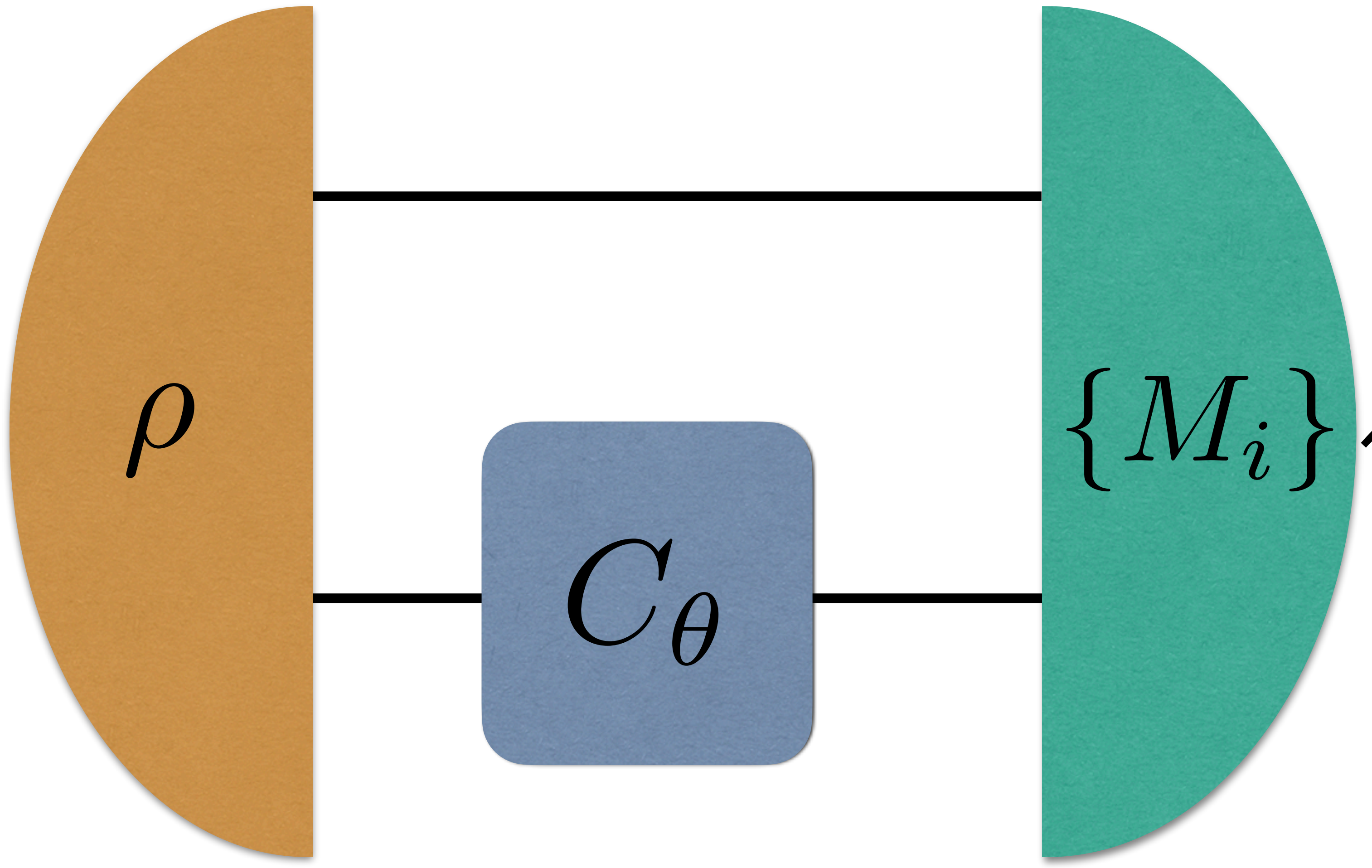

$\sim \quad \{\rho, \{M_i\}, \{\hat{\theta}_i\}\}$

triplet of state, measurement and estimator

$$C_\theta := \sum_{i,j} |i\rangle\langle j| \otimes \mathcal{E}_\theta[|i\rangle\langle j|] \qquad \text{(choi operator)}$$

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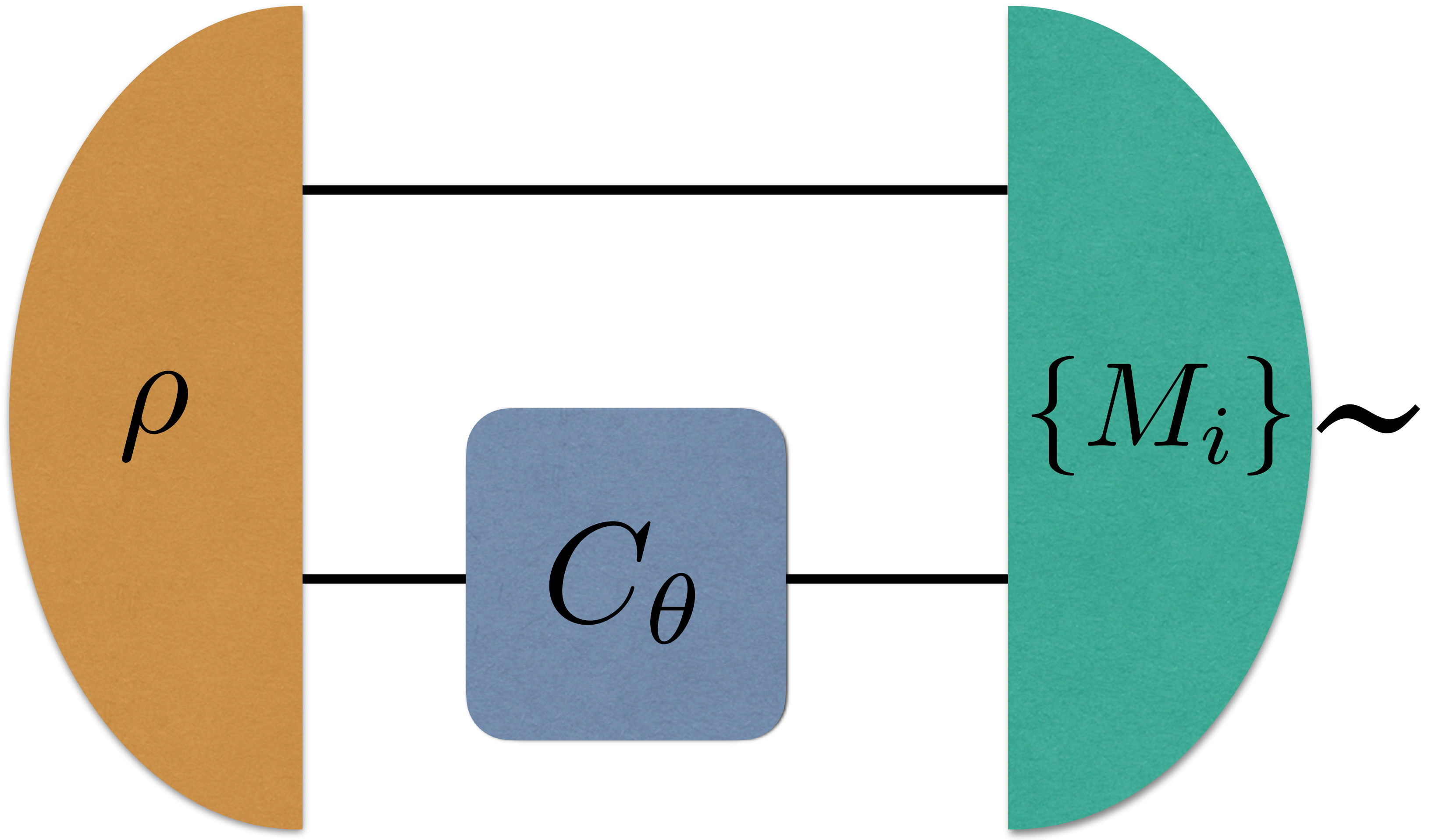

$$\overbrace{\{\rho, \{M_i\}}^{\{T_i\}}, \{\hat{\theta}_i\}\}$$

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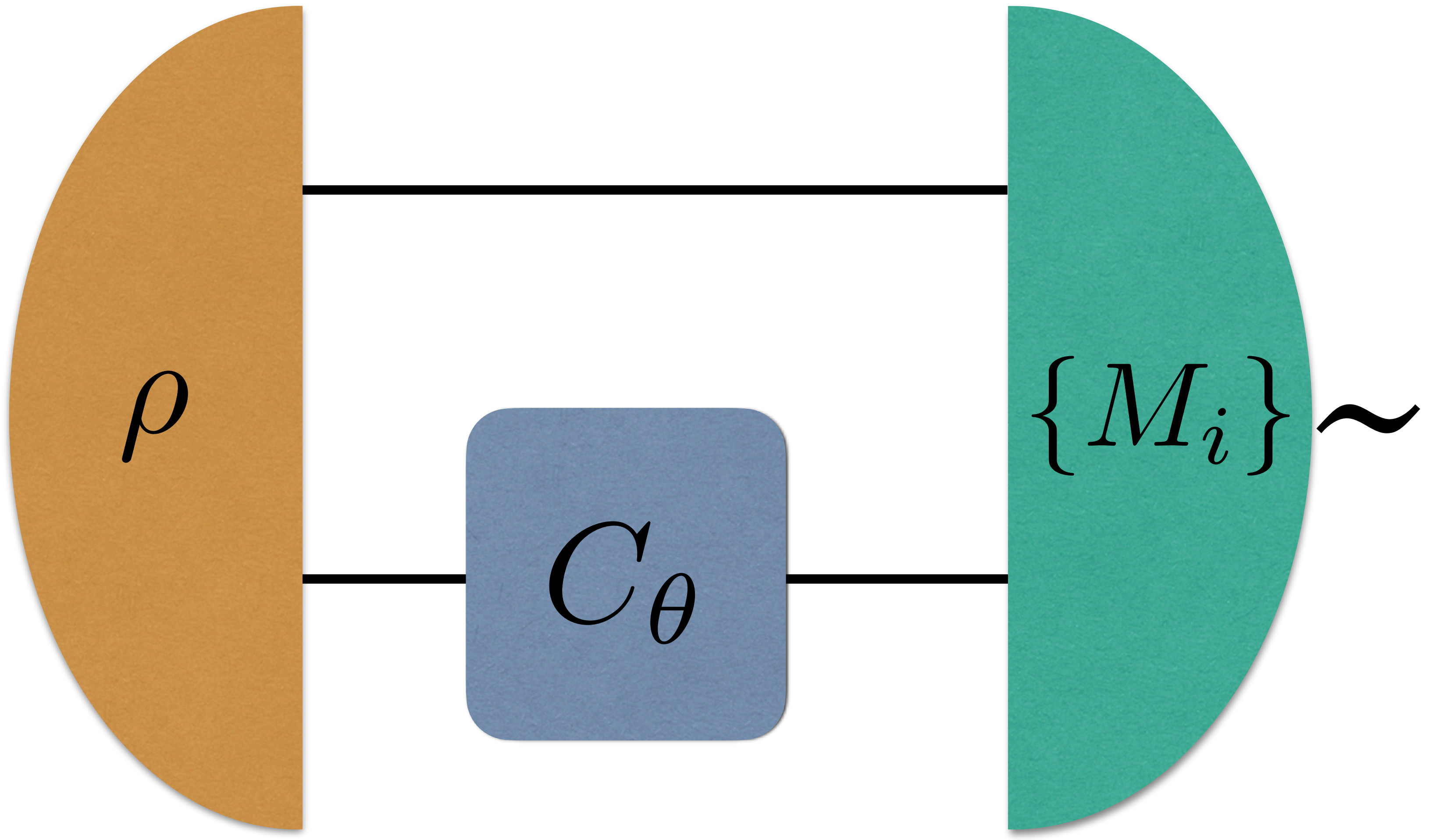

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pair of tester and estimator

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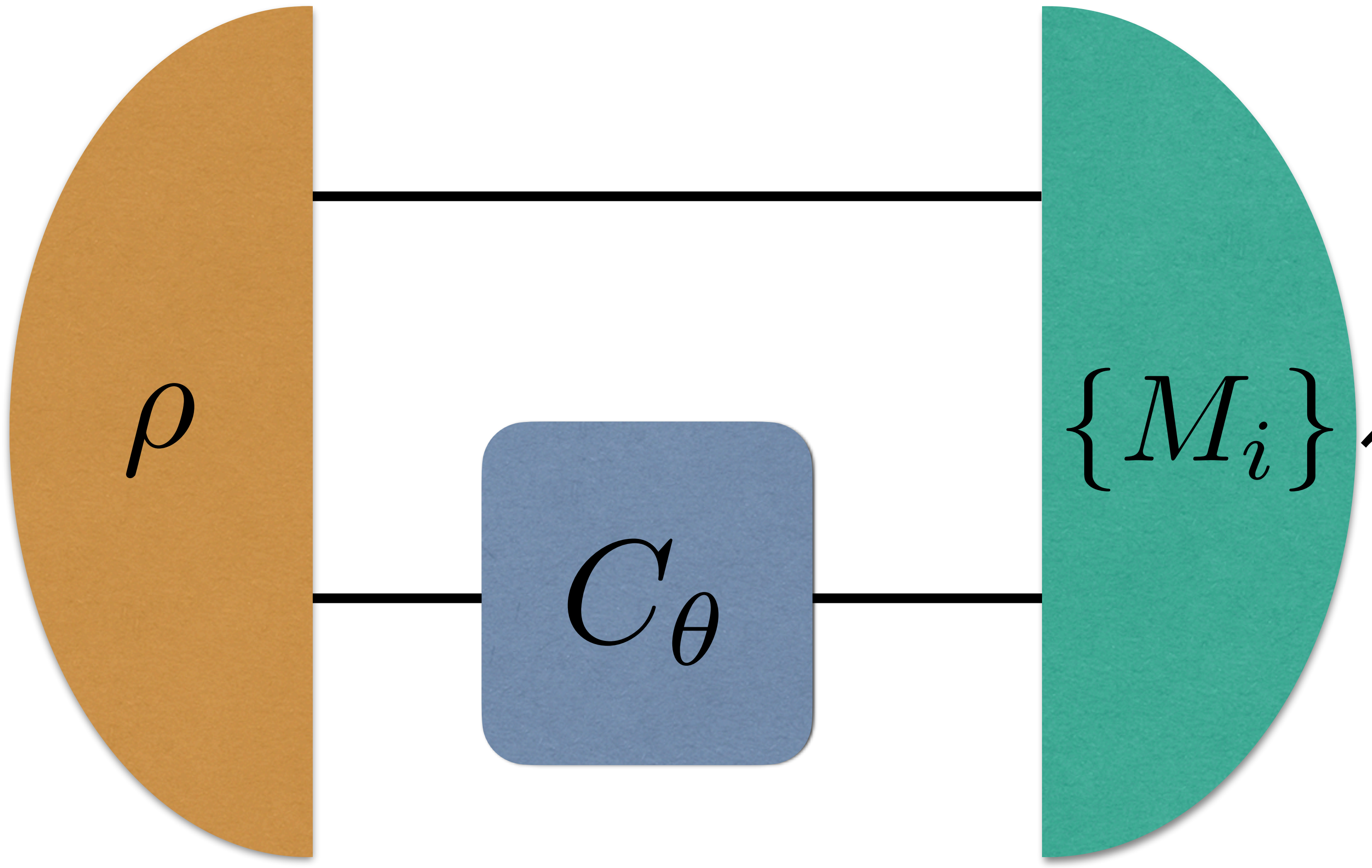

$\sim$

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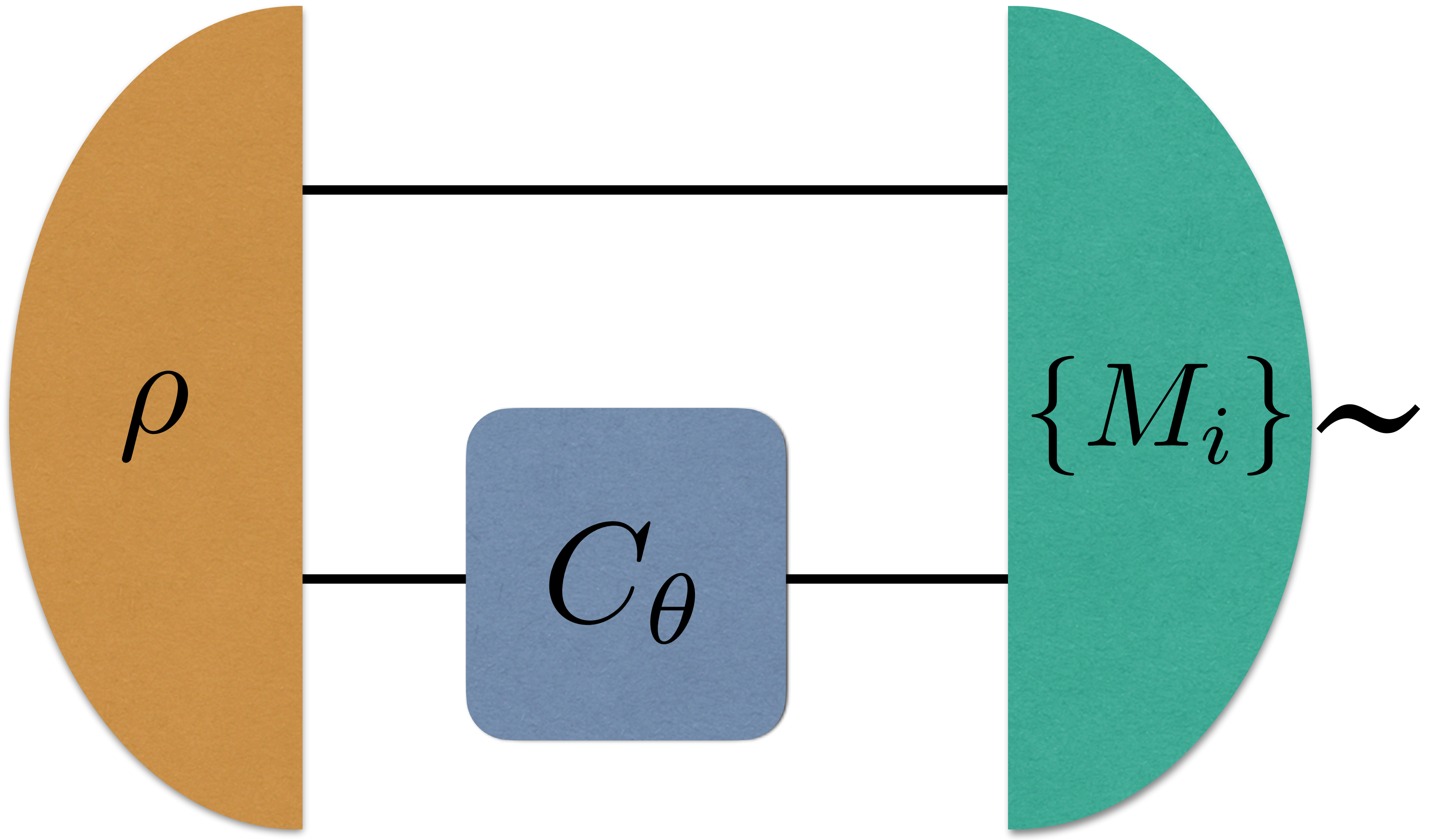
pair of tester and estimator

$$\max_{\{T_i\},\{\hat{\theta}_i\}} \mathcal{S} = \max_{\{T_i\},\{\hat{\theta}_i\}} \sum_{i=1}^{N_O} \operatorname{tr}\left(X(\hat{\theta}_i)\, T_i\right)$$

$$X(\hat{\theta}_i) := \int d\theta\, p(\theta)\, r(\theta,\hat{\theta}_i)\, C_\theta \quad \forall\, i$$

Higher-order operations: quantum testers



$$\{\{T_i\}, \{\hat{\theta}_i\}\}$$

pair of tester and estimator

optimal score: $\mathcal{S}^* := \max_{\{T_i\},\{\hat{\theta}_i\}} \mathcal{S}$

optimal protocol: $\{T_i^*\}, \{\hat{\theta}_i^*\} := \arg \max_{\{T_i\},\{\hat{\theta}_i\}} \mathcal{S}$

our methods

Known optimal estimators (method 2)

for known optimal estimators: $\{\hat{\theta}_i^*\}$

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$$\text{s.t. } T_i \geq 0 \quad \forall\, i$$

$$\sum_i T_i = \sigma \otimes \mathbb{I}$$

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Semidefinite programming (SDP)

Problems

remaining difficulties

- finding the optimal estimators $\{\hat{\theta}_i^*\}$
- computing the integral $X(\hat{\theta}_i) \; = \int d\theta \, p(\theta) \, r(\theta, \hat{\theta}_i) \, C_\theta \quad \forall \, i$

Unknown optimal estimators (method 3)

seesaw method

given $\{X(\hat{\theta}_i)\}$

$$\max_{\{T_i\}} \sum_{i=1}^{N_O} \mathrm{tr}\left(X(\hat{\theta}_i)\, T_i\right)$$

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SDP

given $\{T_i\}$

$$\max_{\{\hat{\theta}_i\}} \sum_{i=1}^{N_O} \mathrm{tr}\left(X(\hat{\theta}_i)\, T_i\right)$$

$$\text{s.t. } \; X(\hat{\theta}_i) = \int d\theta\, p(\theta)\, r(\theta, \hat{\theta}_i)\, C_\theta \quad \forall\, i$$

maybe SDP

maybe gradient descent

often analytical reassignment

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$$\boxed{N_O \leq (d_I \times d_O)^2 \qquad \text{extremal POVMS only}}$$

Approximating the integral

$$X(\hat{\theta}_i) := \int d\theta \, p(\theta) \, r(\theta, \hat{\theta}_i) \, C_\theta \quad \forall \, i \qquad \mapsto \qquad \widetilde{X}(\hat{\theta}_i) := \sum_{k=1}^{N_H} p(\theta_k) \, r(\theta_k, \hat{\theta}_i) \, C_{\theta_k} \quad \forall \, i$$

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Known optimal estimators (method 2, discretized)

for known optimal estimators: $\{\hat{\theta}_i^*\}$

$$\mathcal{S}^* = \max_{\{T_i\}} \sum_{i=1}^{N_O} \mathrm{tr}\left(\widetilde{X}(\hat{\theta}_i)\, T_i\right)$$

$$\text{s.t. } T_i \geq 0 \quad \forall\, i$$

$$\sum_i T_i = \sigma \otimes \mathbb{I}$$

Semidefinite programming (SDP)

Unknown optimal estimators (method 3, discretized)

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N_H	**arbitrarily high**
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Mapping to channel discrimination (method 1)

optimizing estimators and strategy simultaneously

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Direct assignment between estimators and values in the discretization of the parameter

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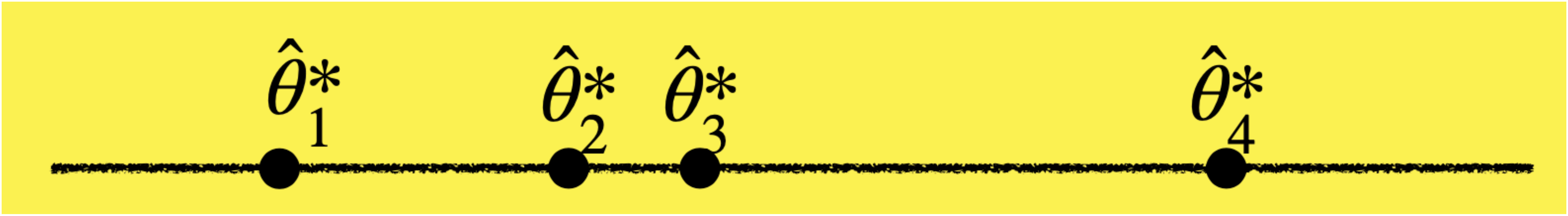

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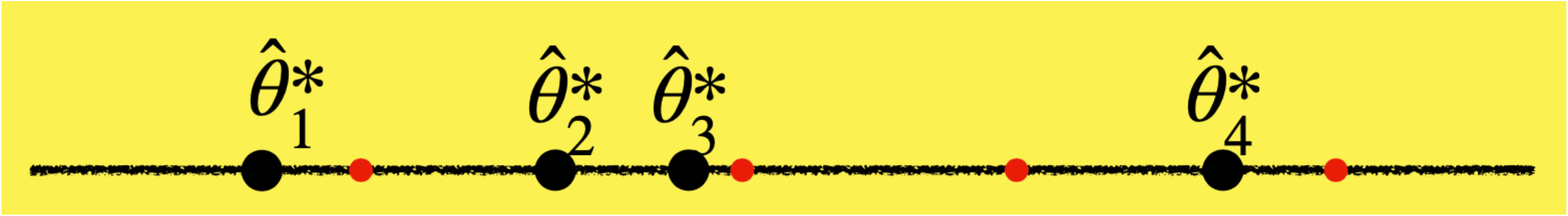
SDP



Mapping to channel discrimination (method 1)

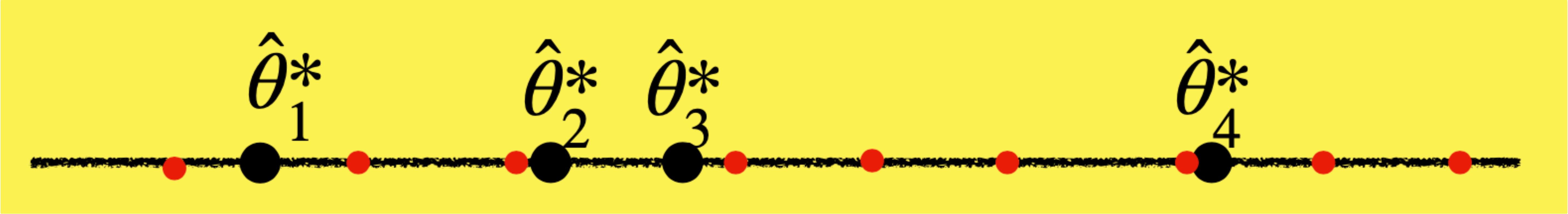
given $\{\theta_i\}$

$$\max_{\{T_i\}} \sum_{i=1}^{N} \mathrm{tr}\left(\widetilde{X}(\theta_i)\, T_i\right)$$

$$\text{s.t. } T_i \geq 0 \quad \forall\, i$$

$$\sum_i T_i = \sigma \otimes \mathbb{I}$$

SDP



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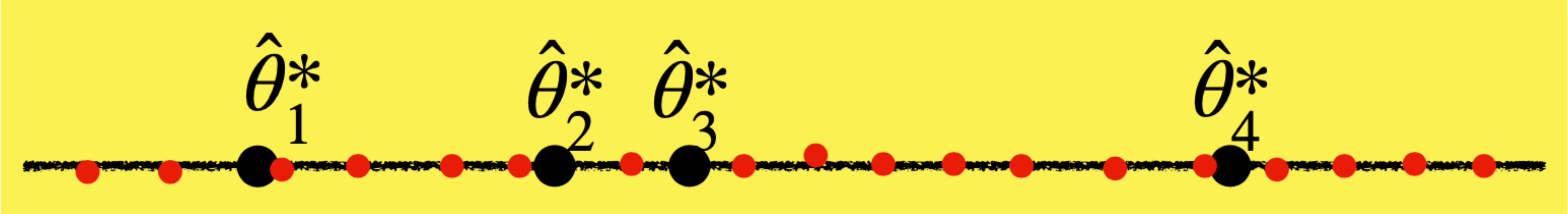

SDP

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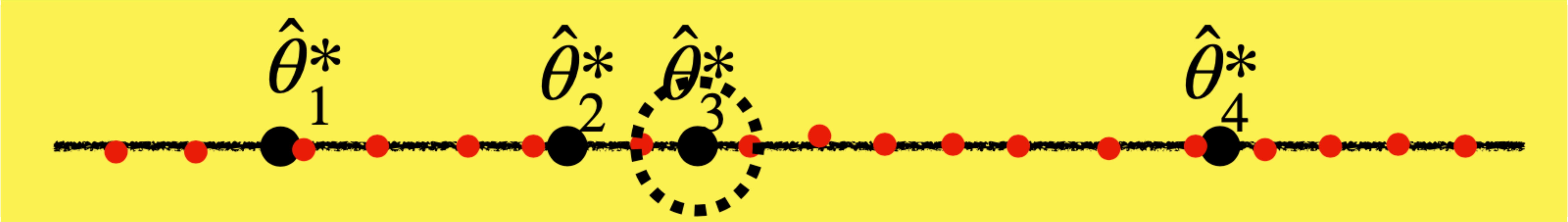
SDP



Mapping to channel discrimination (method 1)

$$\text{given} \quad \{\theta_i\}$$

$$\max_{\{T_i\}} \sum_{i=1}^{N} \operatorname{tr}\left(\widetilde{X}(\theta_i)\, T_i\right)$$

$$\text{s.t. } T_i \geq 0 \quad \forall\ i$$

$$\sum_i T_i = \sigma \otimes \mathbb{I}$$

SDP

$$|\mathcal{S}^* - \widetilde{\mathcal{S}}^*| = \mathcal{O}\left(\frac{1}{N}\right)$$

Our methods

Method 1 (channel discrimination)	Method 2 (known opt estimator)	Method 3 (iterative)
single instance of SDP	single instance of SDP	seesaw
doesn’t require previous knowledge of the estimator	requires previous knowledge of the estimator	doesn’t require previous knowledge of the estimator
converges to optimal solution in the limit	yields exact solution	not guaranteed to converge to optimal solution
may require too many measurement outcomes to be practical	only limited number of outcomes required	only limited number of outcomes required
very good results in practice	very good results in practice even if the estimators are not known to be optimal	very good results in practice

Generalizations

Multi parameter estimation

$$\mathcal{S} = \sum_{i_1=1}^{N_{O_1}} \sum_{i_2=1}^{N_{O_2}} \int d\theta^1 \int d\theta^2 \; p(\theta^1, \theta^2)\, r(\theta^1, \theta^2, \hat{\theta}^1_{i_1}, \hat{\theta}^2_{i_1}) \; \mathrm{tr}\big(C_{\theta^1,\theta^2}\, T_{i_1,i_2}\big).$$

$$\widetilde{\mathcal{S}} = \sum_{i_1=1}^{N_{O_1}} \sum_{i_2=1}^{N_{O_2}} \sum_{k_1=1}^{N_{H_1}} \sum_{k_2=1}^{N_{H_2}} p(\theta^1_{k_1}, \theta^2_{k_2})\, r(\theta^1_{k_1}, \theta^2_{k_2}, \hat{\theta}^1_{i_1}, \hat{\theta}^2_{i_2}) \; \mathrm{tr}\Big(C_{\theta^1_{k_1},\theta^2_{k_2}}\, T_{i_1,i_2}\Big).$$

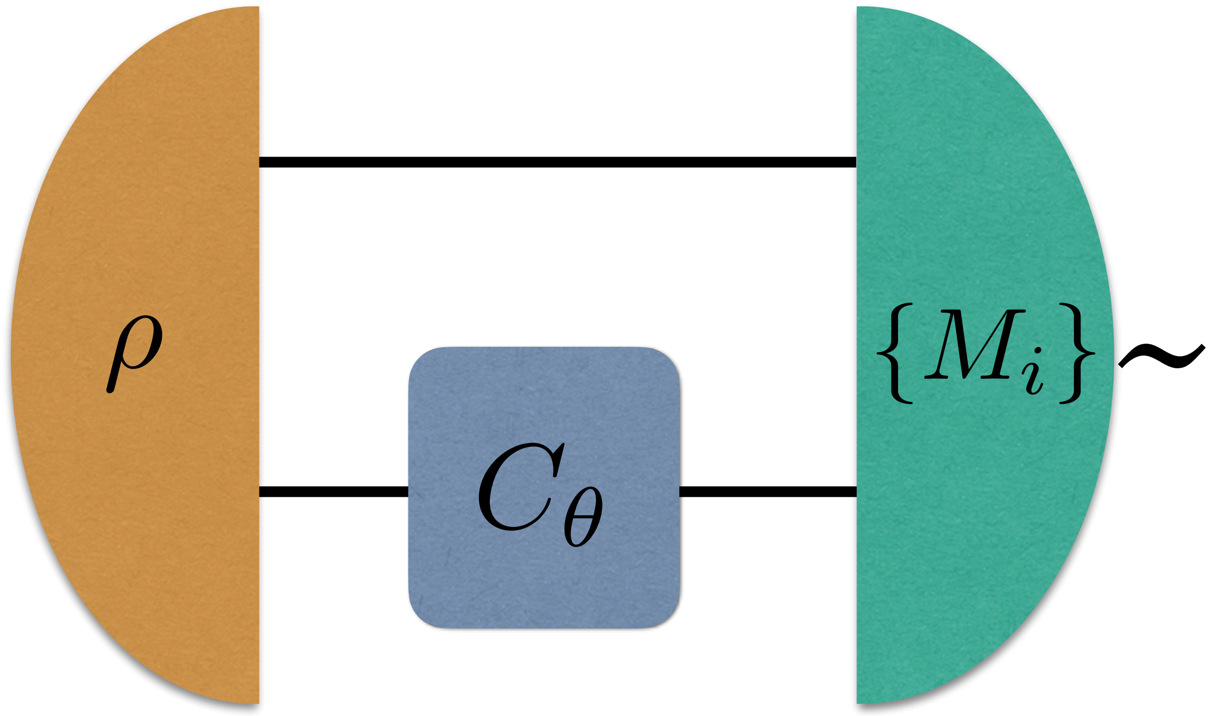
$(i_1, i_2) \mapsto i$, where $i \in \{1, \dots, N_O\}$, $N_O = N_{O_1} N_{O_2}$

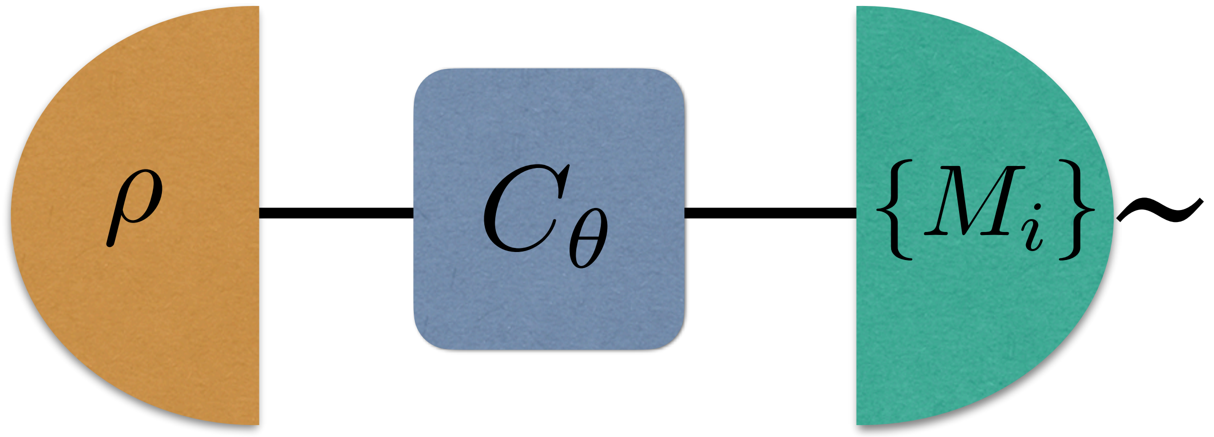
$(k_1, k_2) \mapsto k$, where $k \in \{1, \dots, N_H\}$ $\quad N_H = N_{H_1} N_{H_2}$

$$\widetilde{\mathcal{S}} = \sum_{i=1}^{N_O} \sum_{k=1}^{N_H} p(\vec{\theta}_k)\, r(\vec{\theta}_k, \hat{\vec{\theta}}_i) \; \mathrm{tr}\Big(C_{\vec{\theta}_k}\, T_i\Big)$$

Generalizations

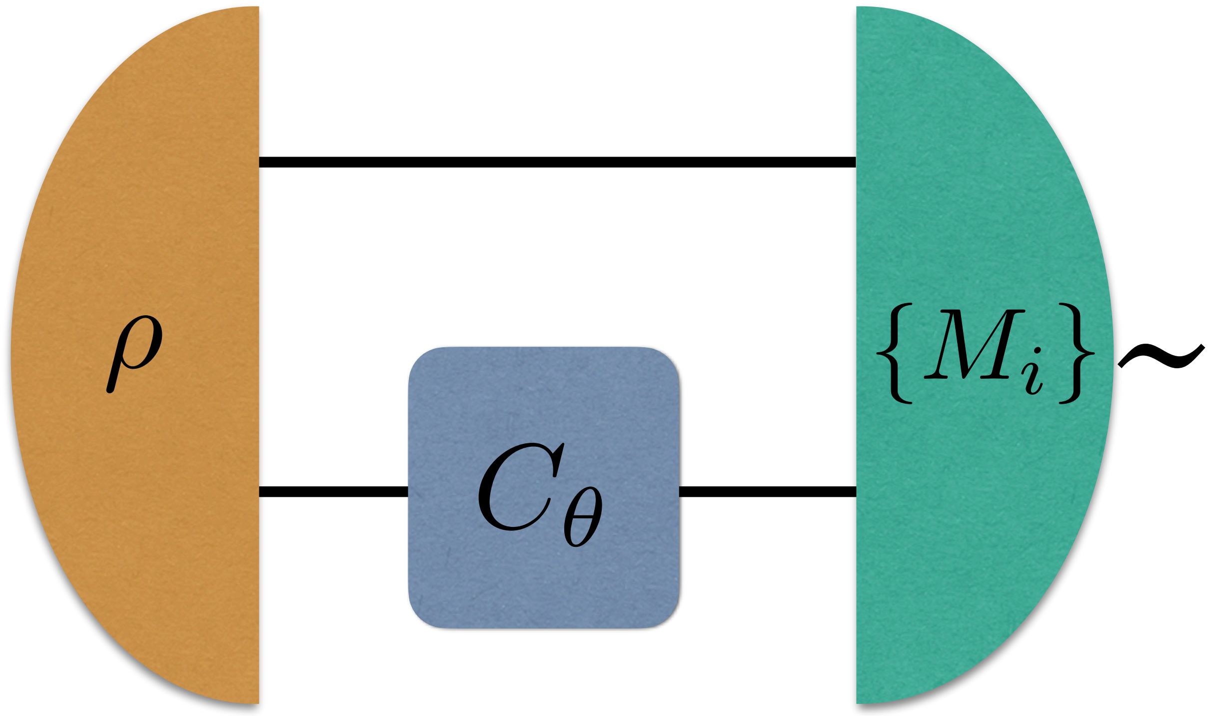
Outer approximation for strategies without memory/entanglement

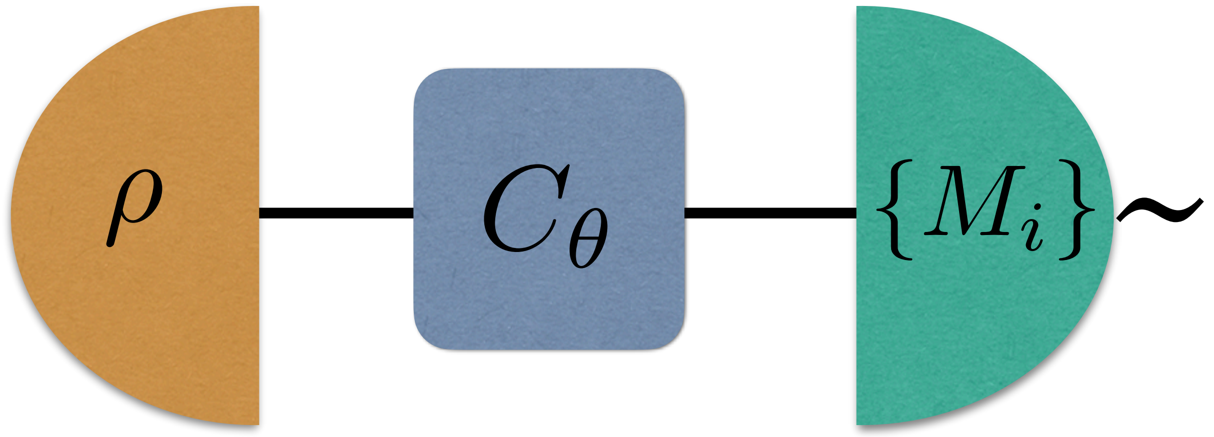




Generalizations

Outer approximation for strategies without memory/entanglement





$$T_i = \mathrm{Tr}_{\mathrm{aux}}\left[(\rho^{\mathrm{in,aux}} \otimes \mathbb{I}^{\mathrm{out}})(\mathbb{I}^{\mathrm{in}} \otimes (M_i^{\mathrm{aux,out}})^{T_{\mathrm{out}}})\right]$$

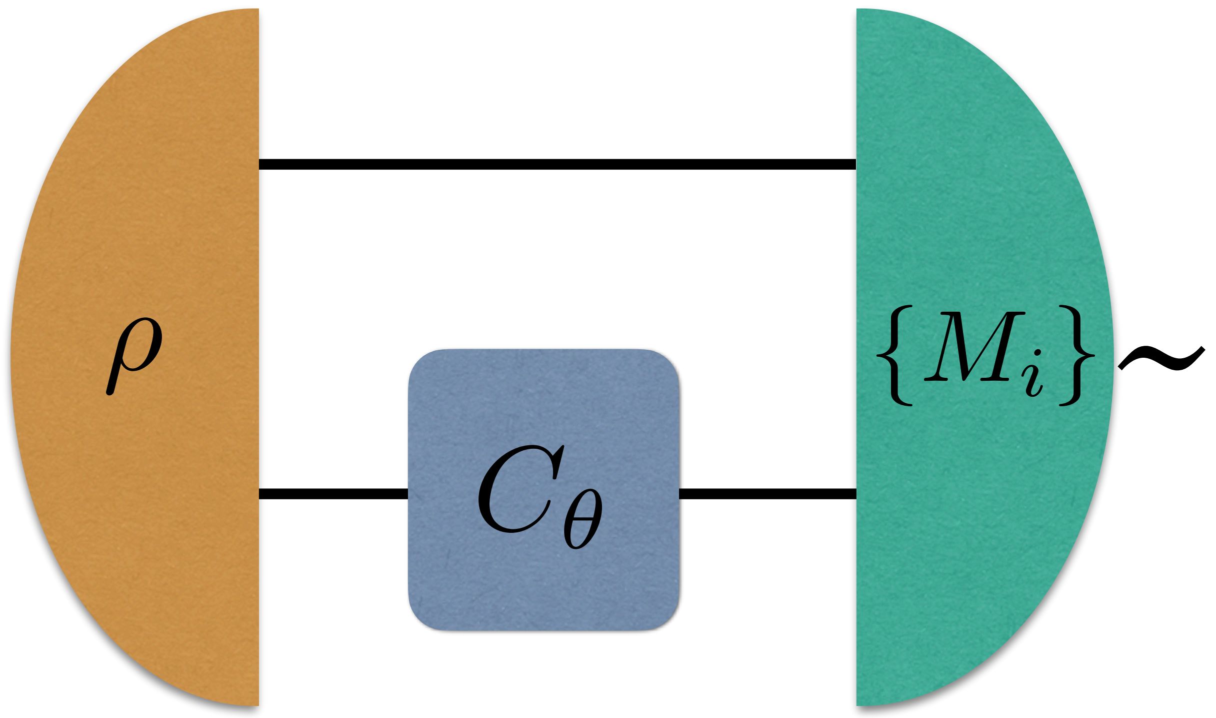
$$T_i = \rho^{\mathrm{in}} \otimes M_i^{\mathrm{out}}$$

$$T_i \geq 0 \quad \forall i$$

$$\sum_i T_i = \sigma \otimes \mathbb{1}^O$$

Generalizations

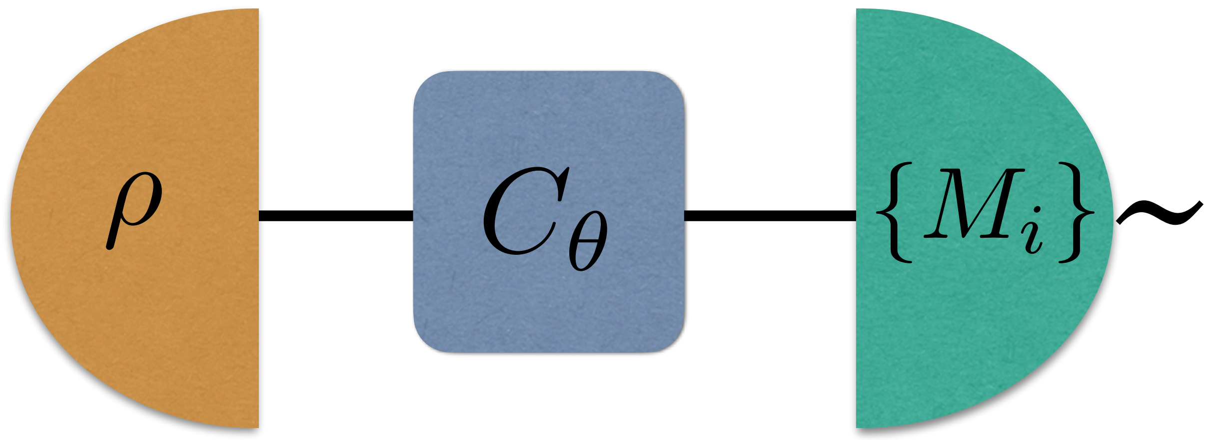
Outer approximation for strategies without memory/entanglement



$\supset$

$$T_i \geq 0 \quad \forall i$$
$$\sum_i T_i = \sigma \otimes \mathbb{1}^O$$
$$(T_i)^{\mathrm{T}_I} \geq 0 \quad \forall i$$

$\supset$



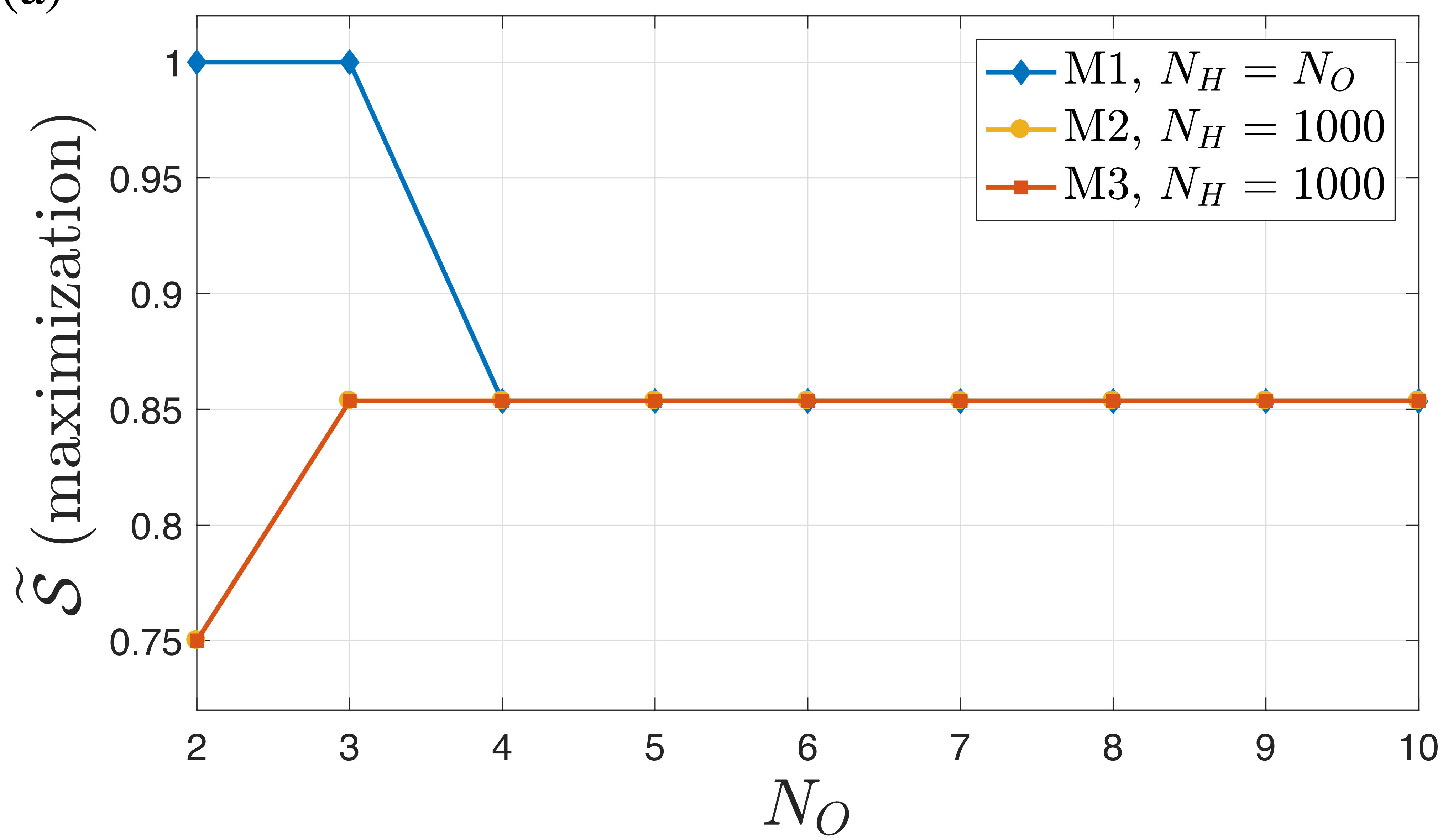
PPT

SDP | **SDP** | **not SDP**

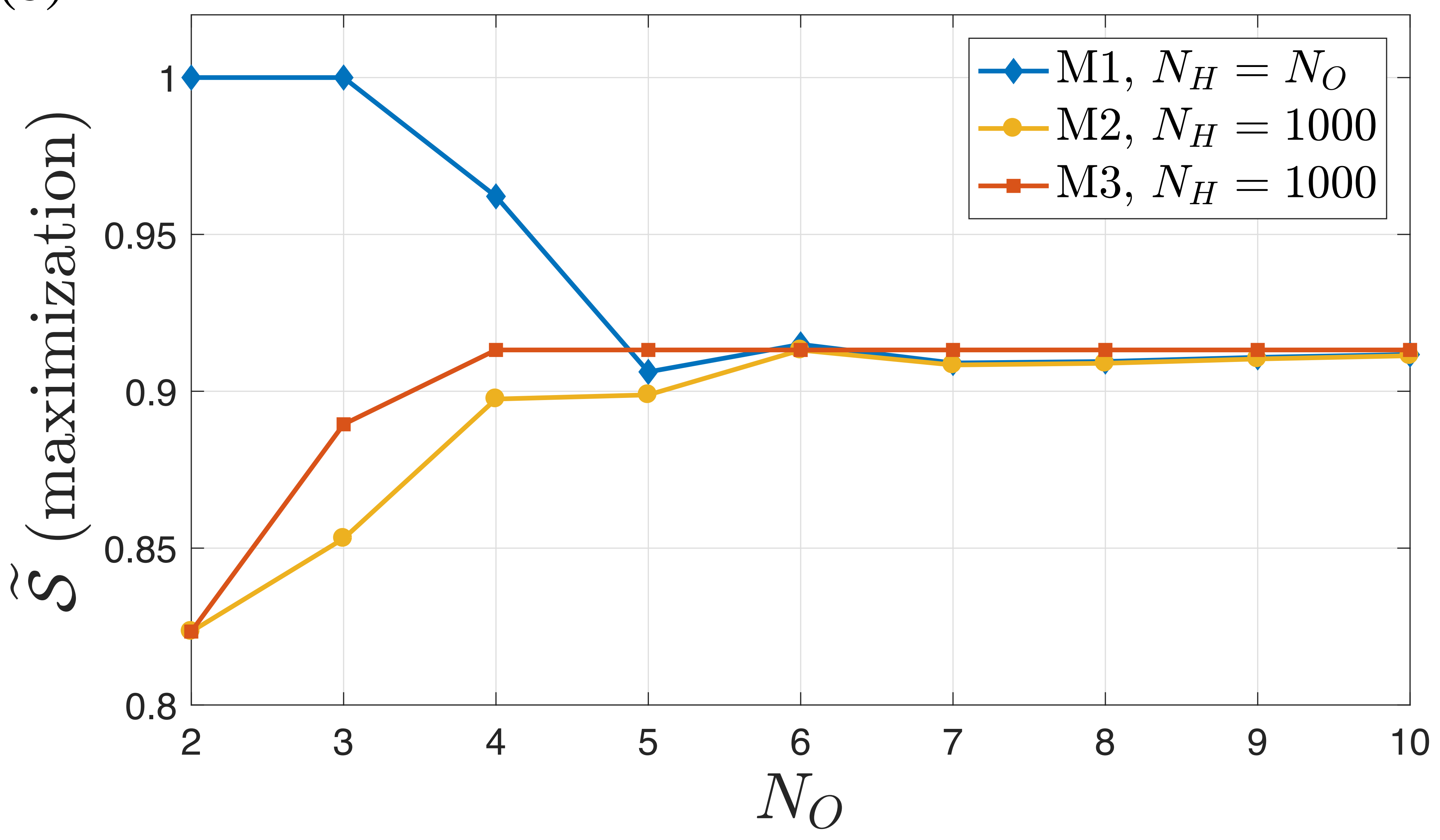
our case studies

Case study 1: phase estimation

(a)



(b)



channel (two-qubit):

$$\mathcal{E}_\theta[\rho] = U_\theta \rho U_\theta^\dagger, \quad U_\theta = e^{-i\theta S_z}, \quad S_z = \frac{1}{2}\sum_{i=0}^{d-1} i|i\rangle\langle i|$$

reward function:

$$r(\theta, \hat{\theta}_i) = \cos^2\left(\frac{\theta - \hat{\theta}_i}{2}\right)$$

prior:

(a) uniform

(b) gaussian

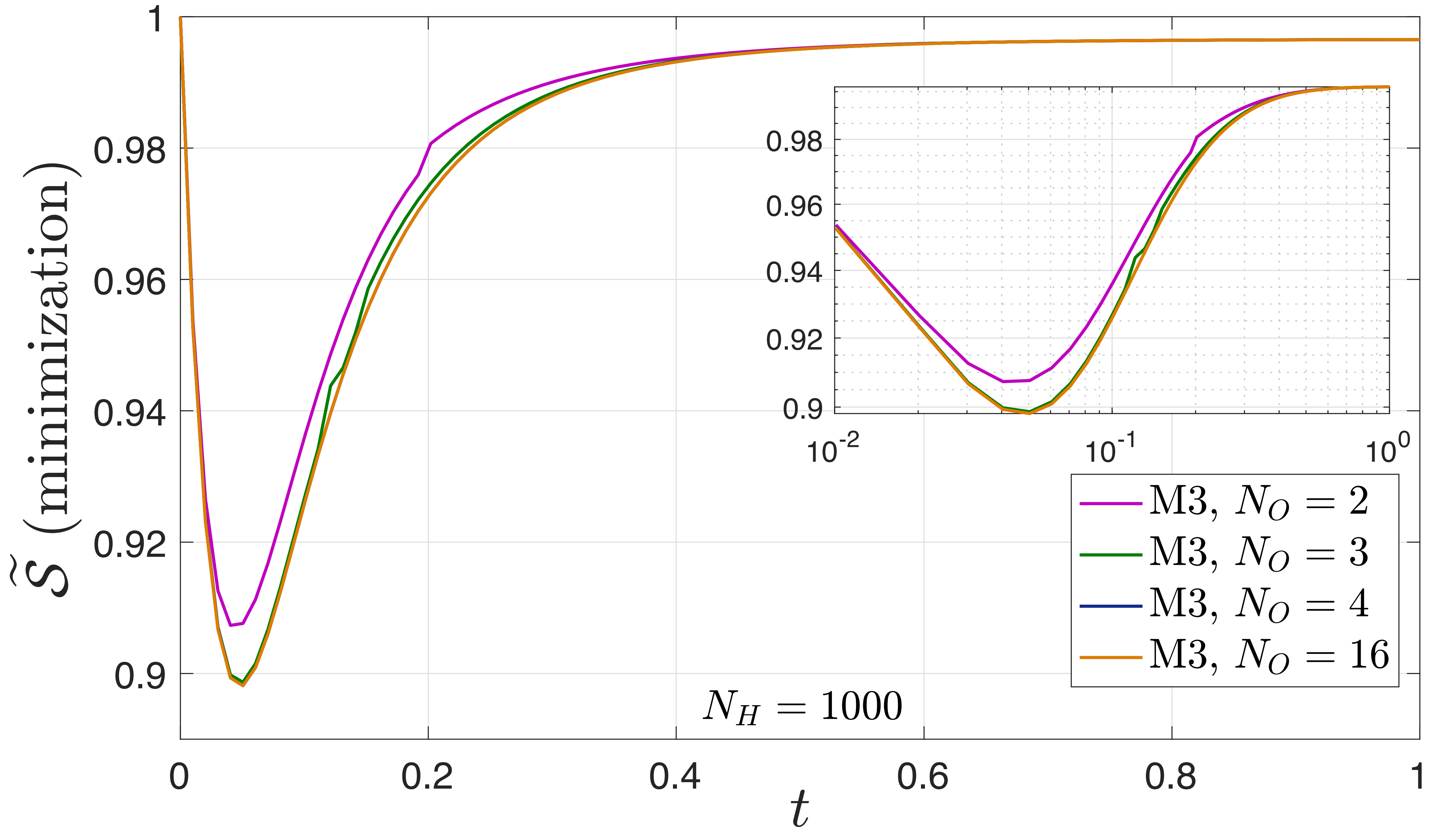
estimator update:

$$\hat{\theta}_i^* = \begin{cases} \arctan\left(\frac{\langle\sin(\theta_k)\rangle^{(i)}}{\langle\cos(\theta_k)\rangle^{(i)}}\right) & \langle\cos(\theta_k)\rangle^{(i)} \geq 0 \\ \arctan\left(\frac{\langle\sin(\theta_k)\rangle^{(i)}}{\langle\cos(\theta_k)\rangle^{(i)}}\right) + \pi & \langle\cos(\theta_k)\rangle^{(i)} < 0 \end{cases},$$

where

$$\langle f(\theta_k)\rangle^{(i)} = \sum_k p(\theta_k|i) f(\theta_k) = \sum_k \frac{p(\theta_k)\,\mathrm{tr}(C_{\theta_k} T_i)}{p(i)} f(\theta_k)$$

Case study 2: thermometry



channel (non-unitary):

$$\dot{\rho}^p_\theta(t) = -i[H, \rho^p_\theta(t)] + \Gamma_{\mathrm{in}}\mathcal{D}[\sigma_+]\rho^p_\theta(t) + \Gamma_{\mathrm{out}}\mathcal{D}[\sigma_-]\rho^p_\theta(t)$$

reward function:

$$r(\theta, \hat{\theta}_i) = (\theta - \hat{\theta}_i)^2$$

prior:

uniform

estimator update:

$$N_O \leq d_I \times d_O$$

$$\hat{\theta}^*_i = \langle \theta_k \rangle^{(i)}$$

where

$$\langle f(\theta_k) \rangle^{(i)} = \sum_k p(\theta_k|i) f(\theta_k) = \sum_k \frac{p(\theta_k)\,\mathrm{tr}(C_{\theta_k} T_i)}{p(i)} f(\theta_k)$$

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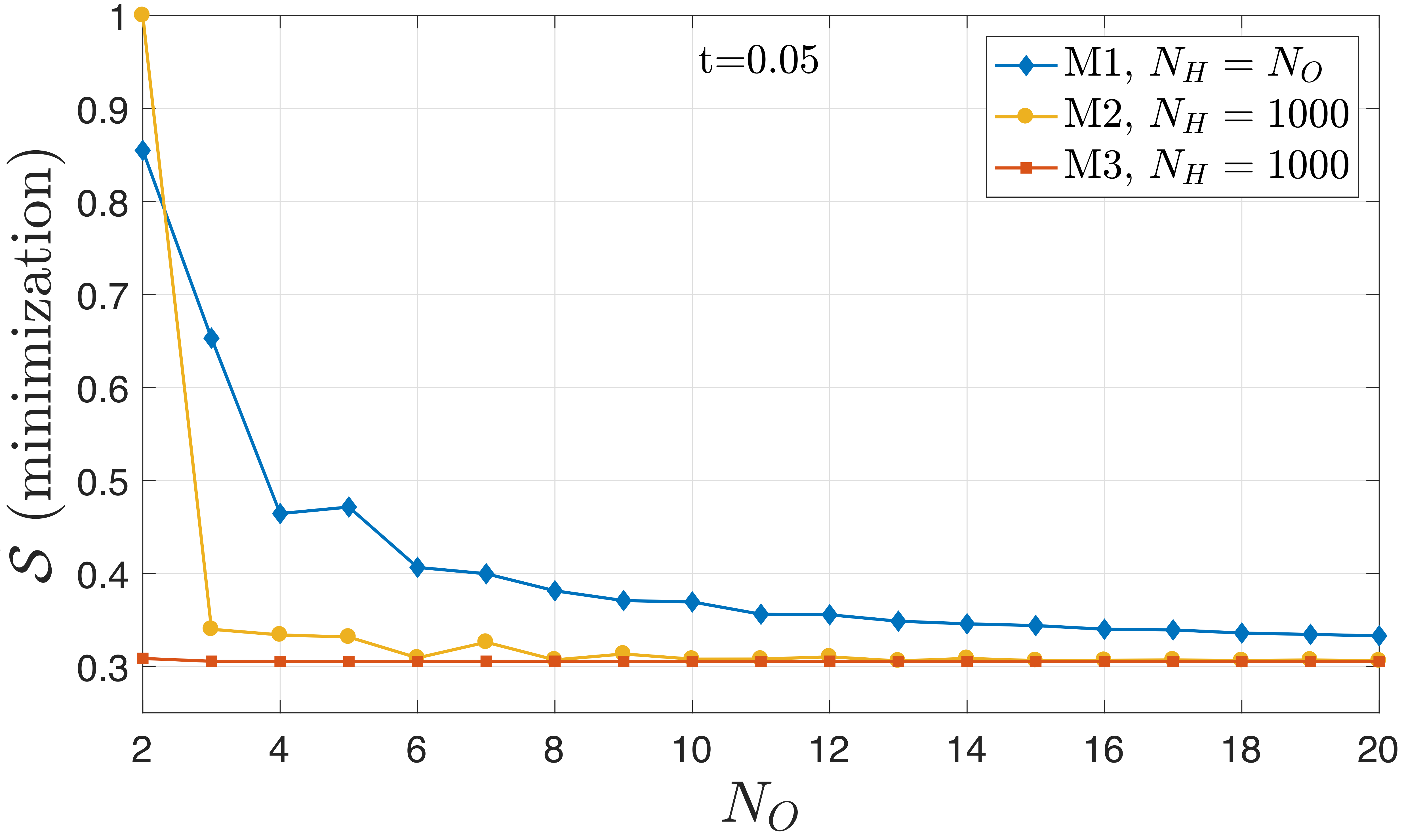

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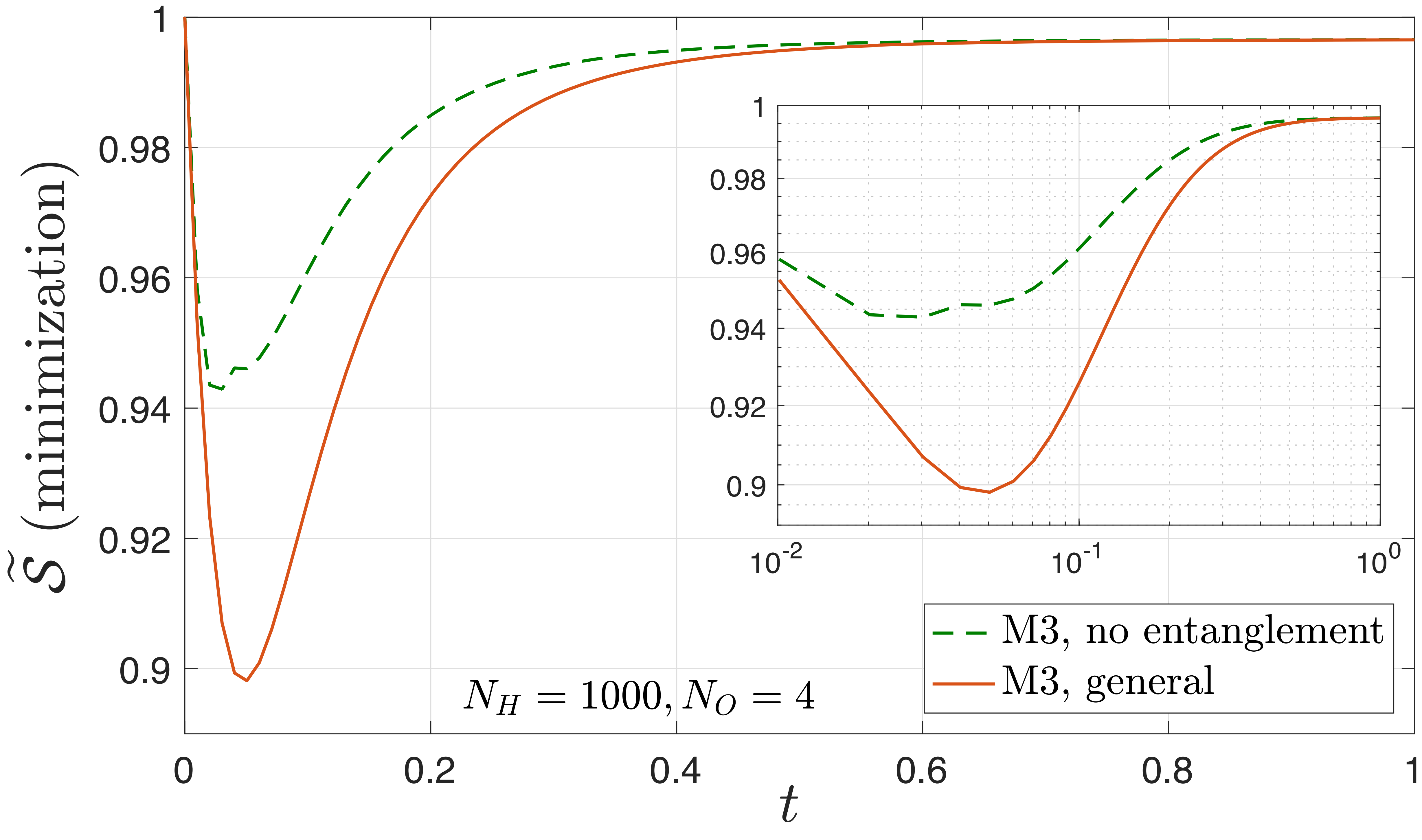

usefulness of entanglement

channel (non-unitary):

$$\dot{\rho}^p_\theta(t) = -i[H, \rho^p_\theta(t)] + \Gamma_{\text{in}}\mathcal{D}[\sigma_+]\rho^p_\theta(t) + \Gamma_{\text{out}}\mathcal{D}[\sigma_-]\rho^p_\theta(t)$$

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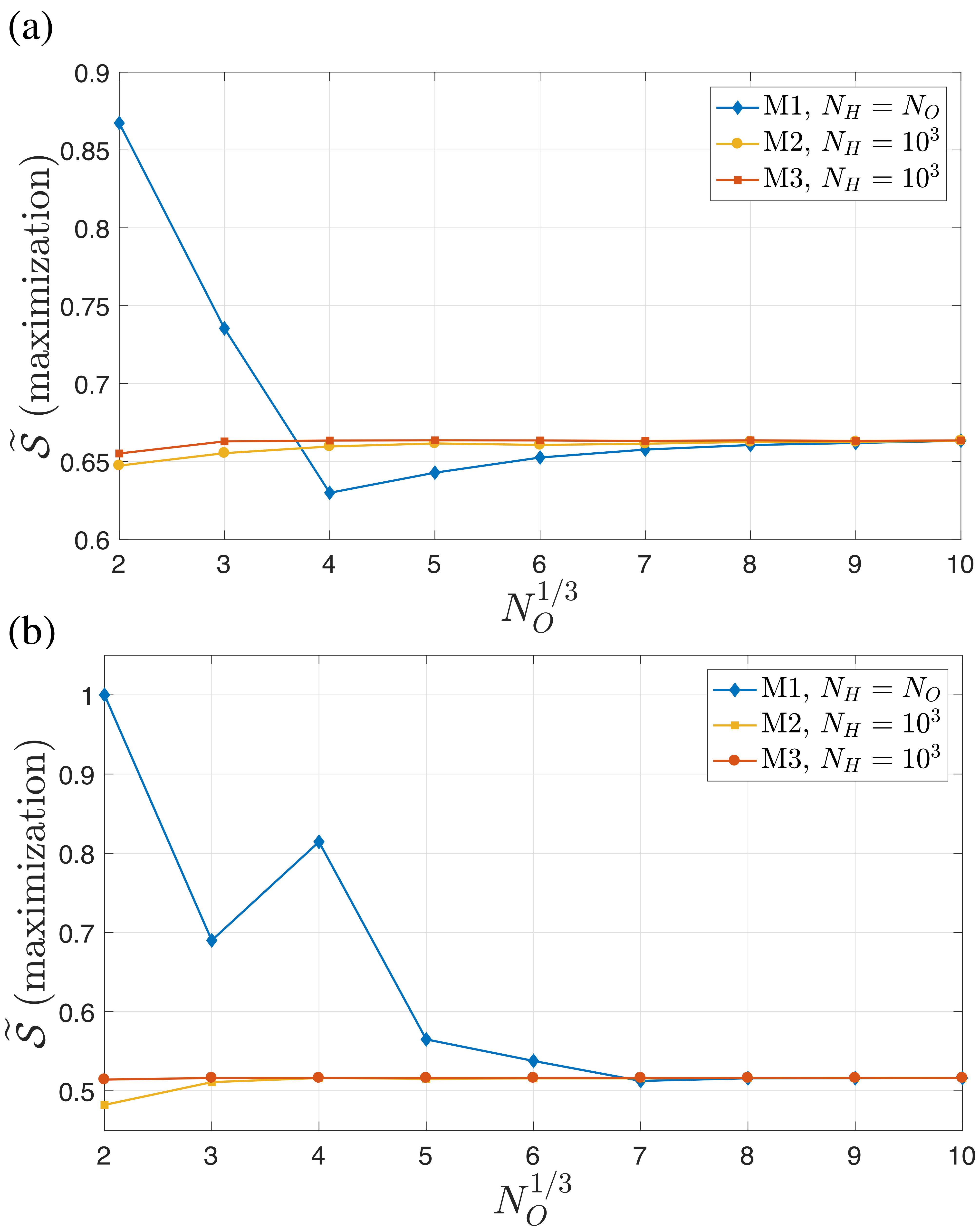
Case study 3: SU(2) estimation



$$\theta := (\theta_x, \theta_y, \theta_z)$$

channel (multi-parameter):

$$\mathcal{E}_\theta[\rho] = U_\theta \rho U_\theta^\dagger, \quad U_\theta = e^{-i(\theta_x \sigma_x + \theta_y \sigma_y + \theta_z \sigma_z)}$$

reward function:

$$r(\theta, \hat{\theta}_i) = \frac{1}{d^2} \left| \langle\langle U_\theta | U_{\hat{\theta}_i} \rangle\rangle \right|^2$$

prior:

(a) uniform

(b) gaussian

estimator update:

gradient descend

Summary and outlook

- **Good approximations** for single-shot Bayesian parameter estimation using SDPs and higher-order operations
- **Single and multiple** parameters
- General strategies or restriction to **no memory/entanglement**
- **Arbitrary** channel encoding, **arbitrary** prior, **arbitrary** cost function

Future:

- Extension to multiple copies, different classes of strategies

Thank you!

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Extra