Qalypso 2026 Summer School

Semidefinite Programming as a Tool for Quantum Information

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How to use SDPs

Is your problem an SDP?

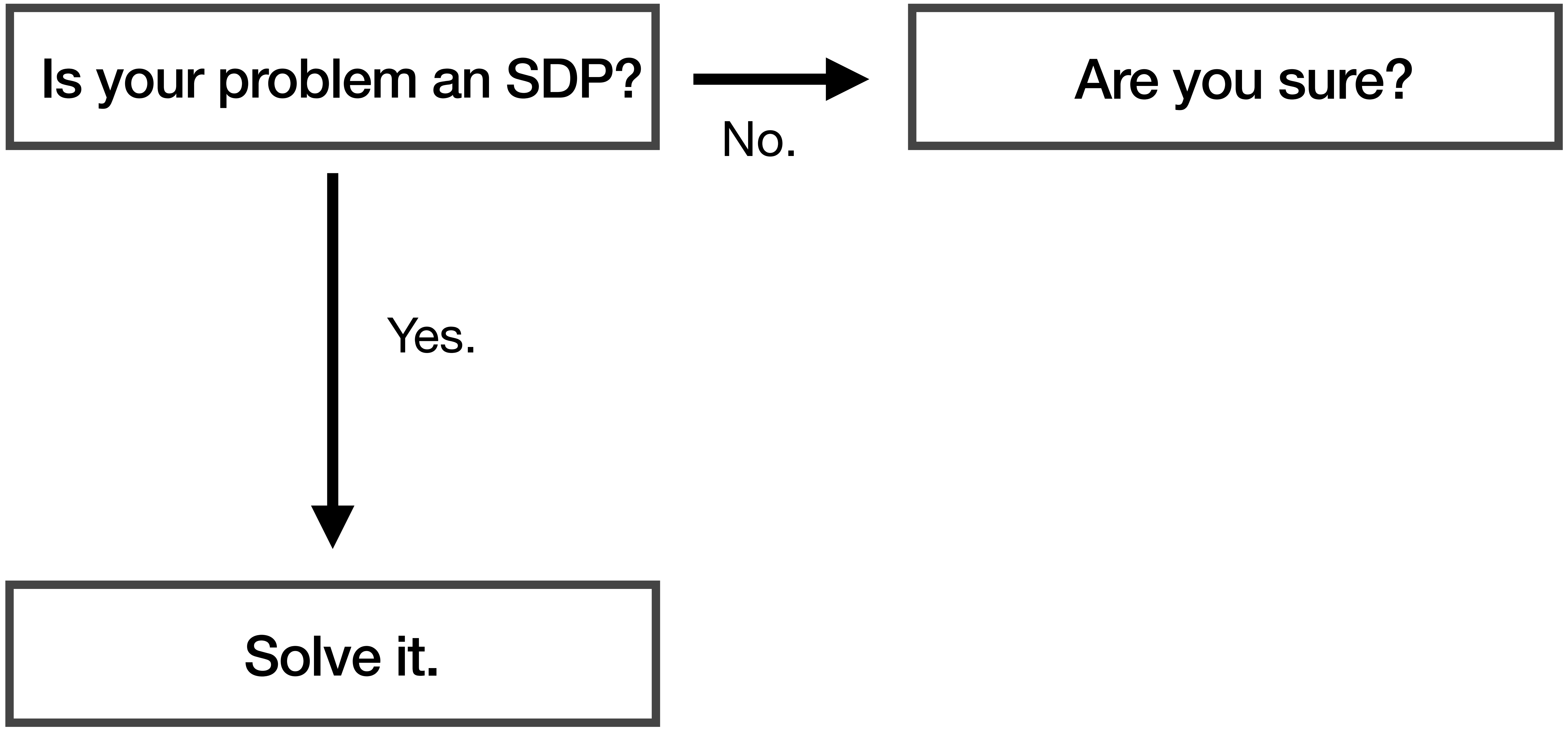
How to use SDPs

Is your problem an SDP?

Yes.

Solve it.

How to use SDPs



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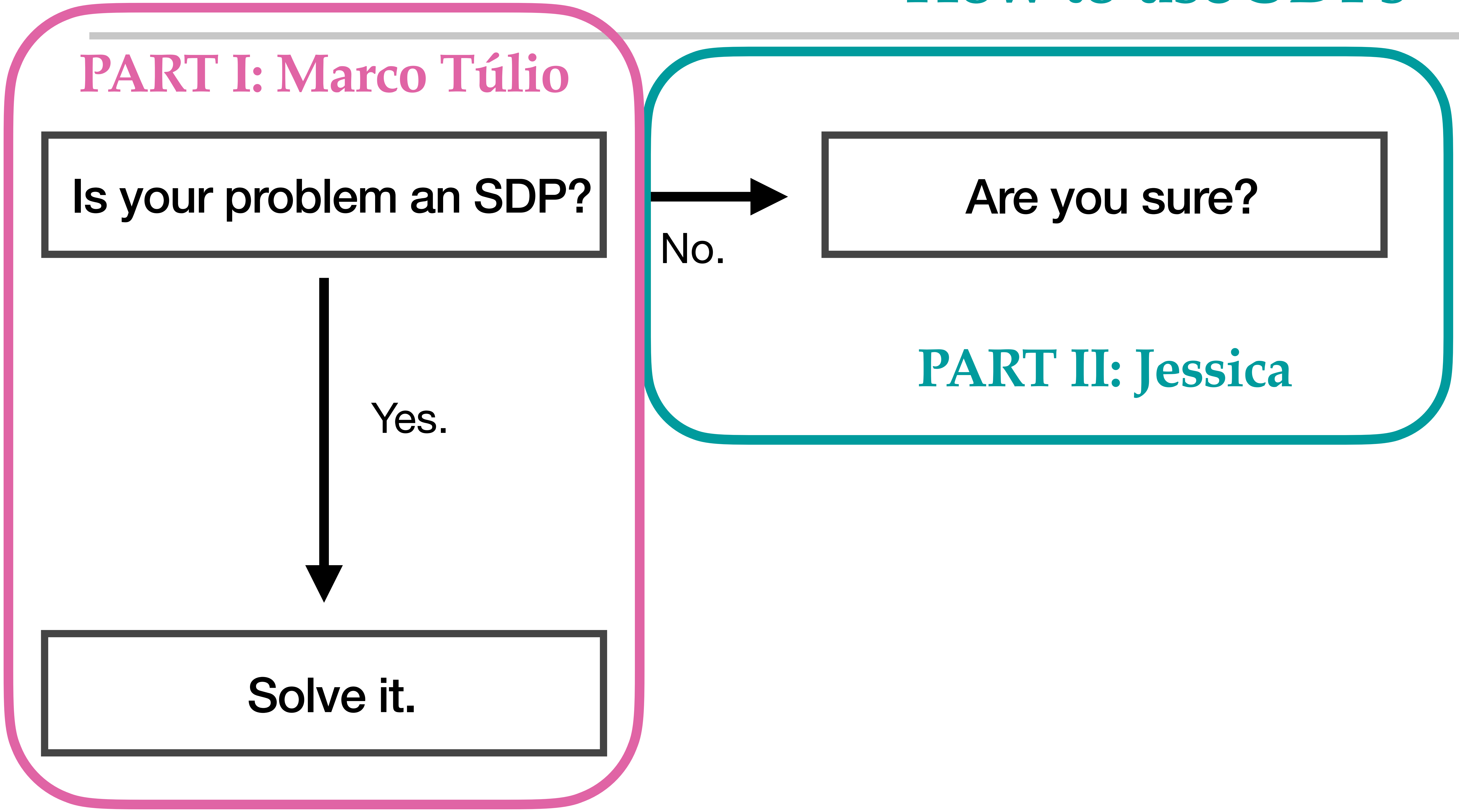

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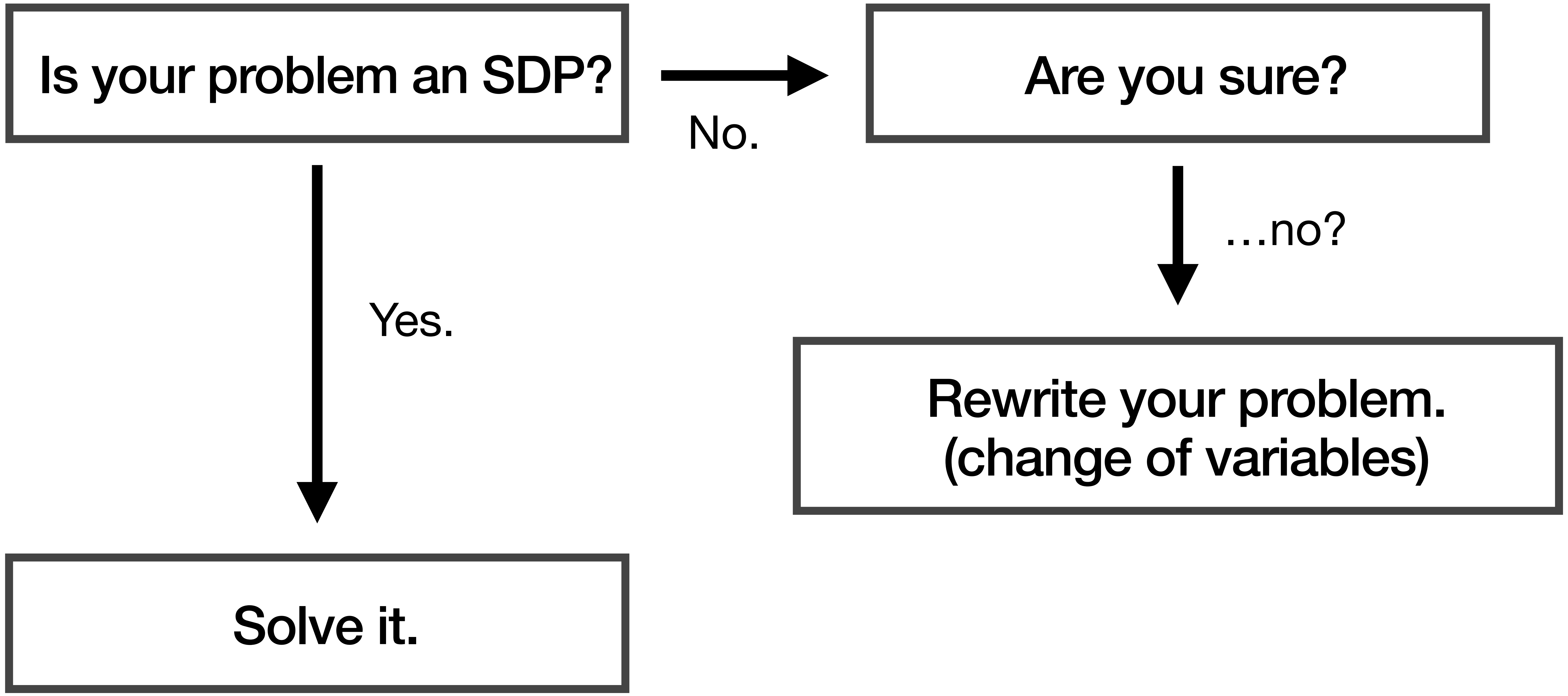

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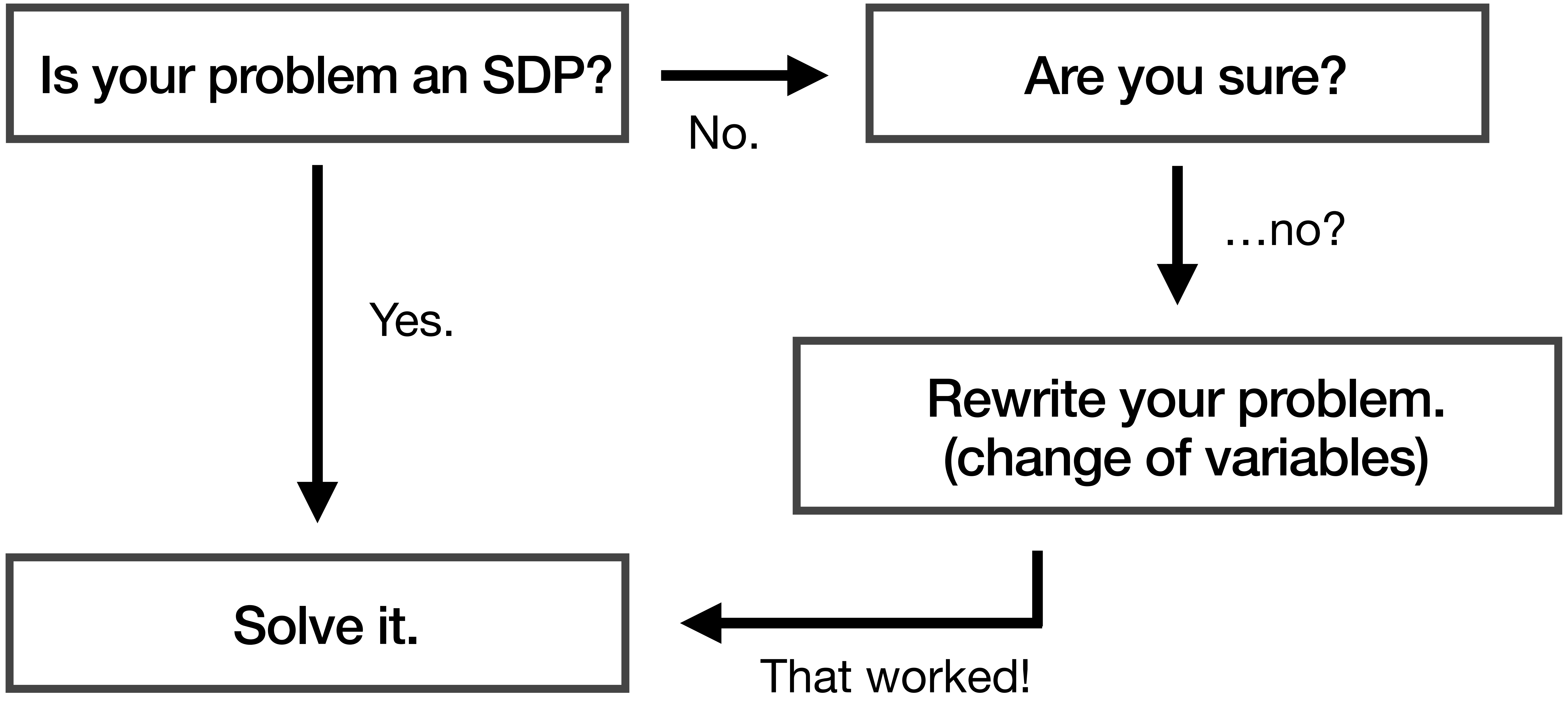

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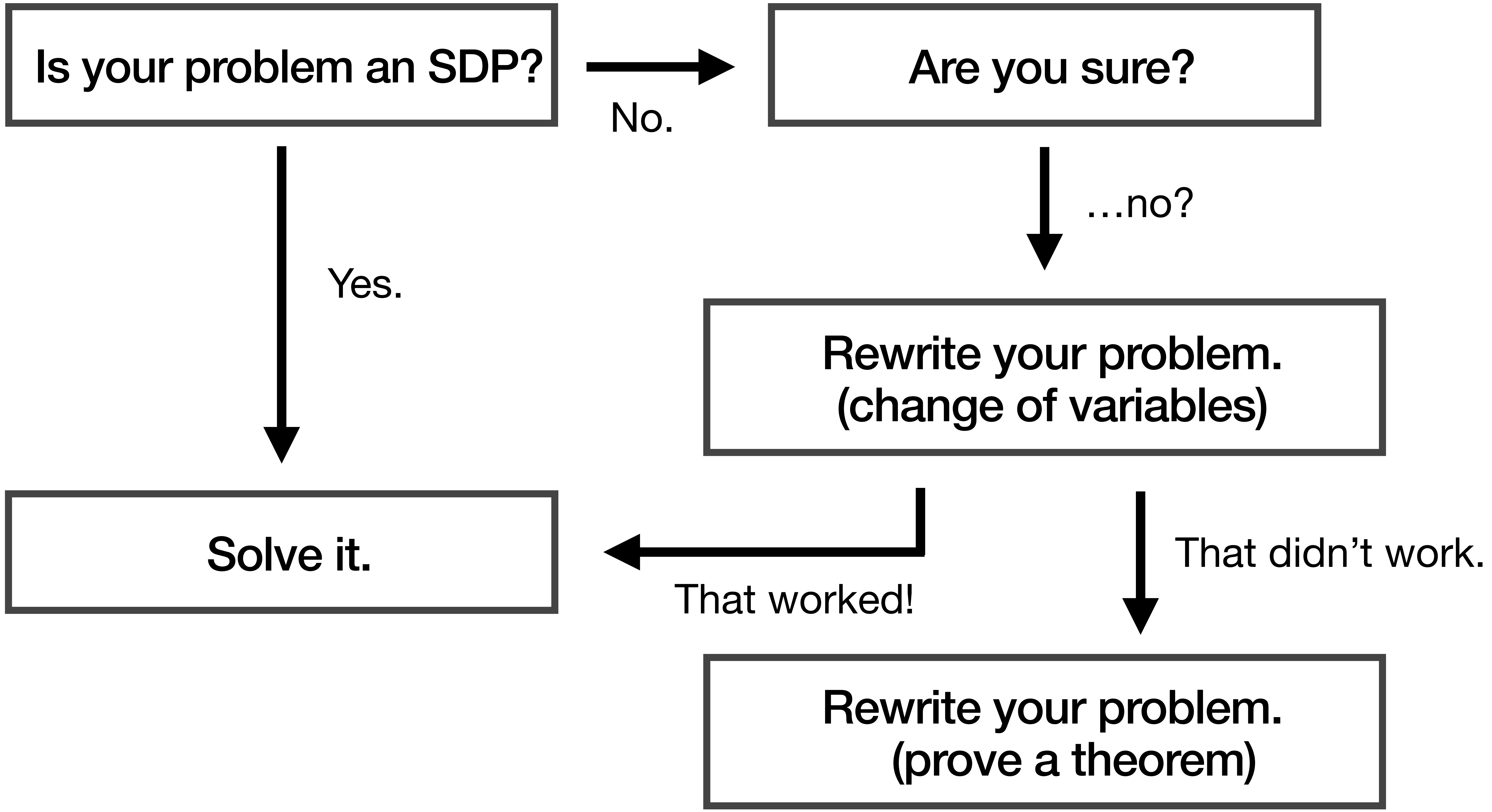

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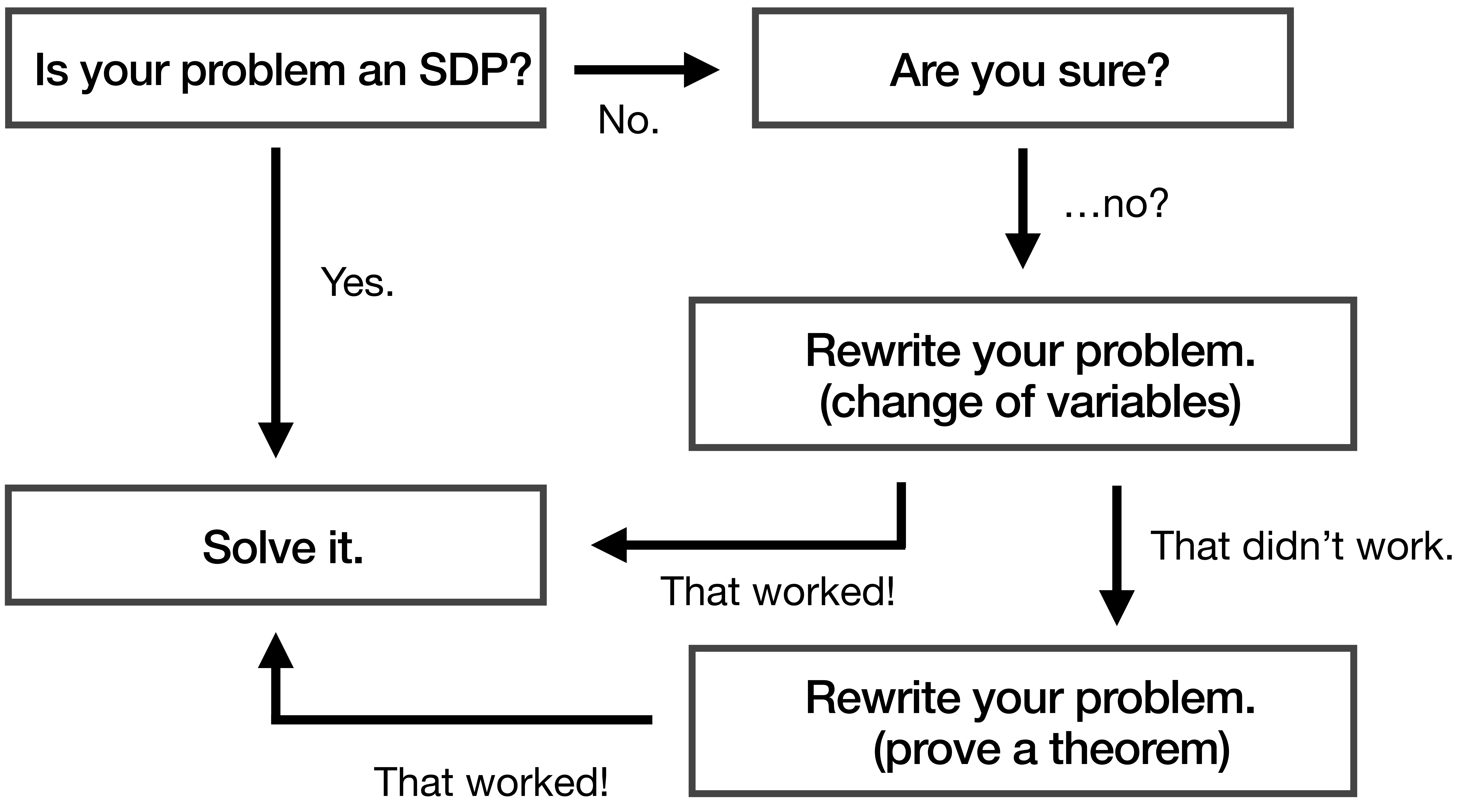

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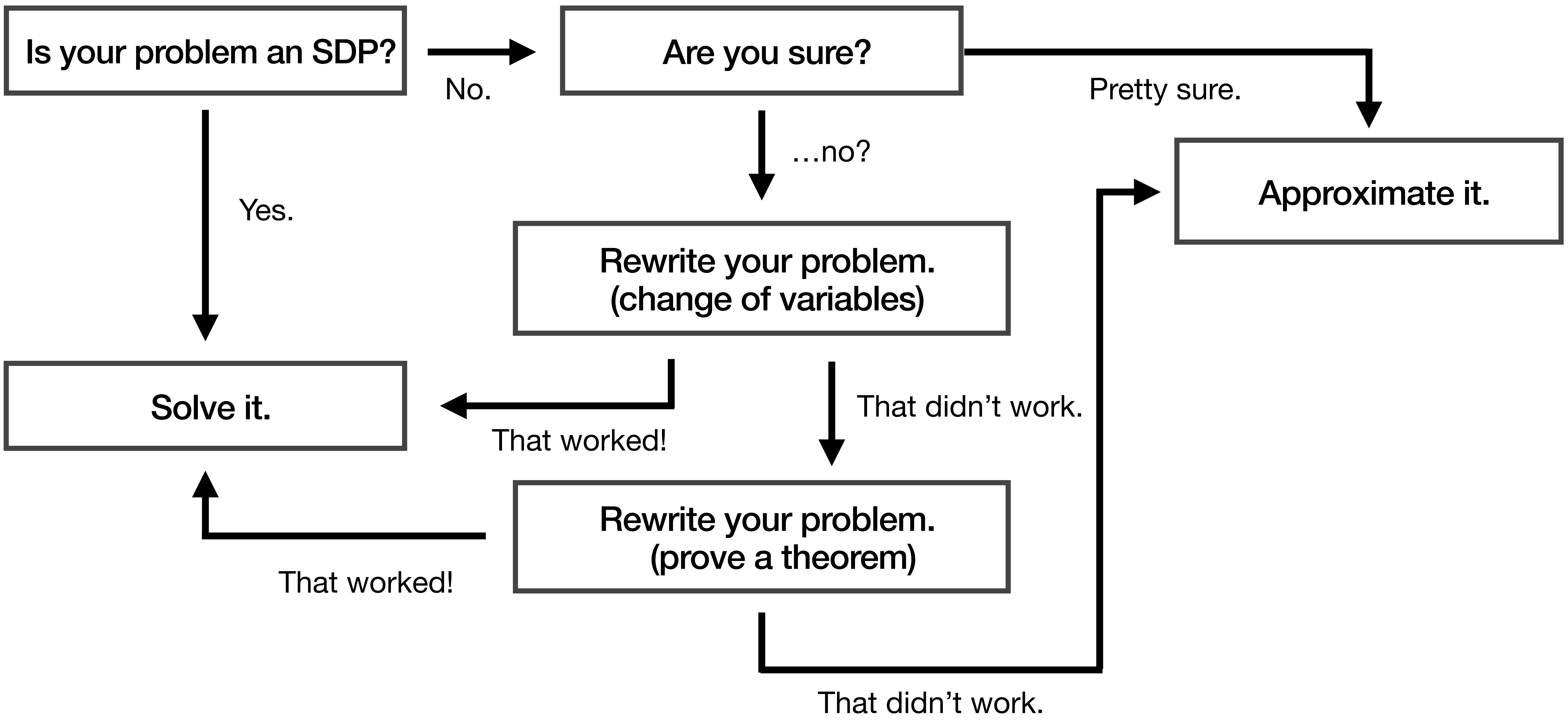

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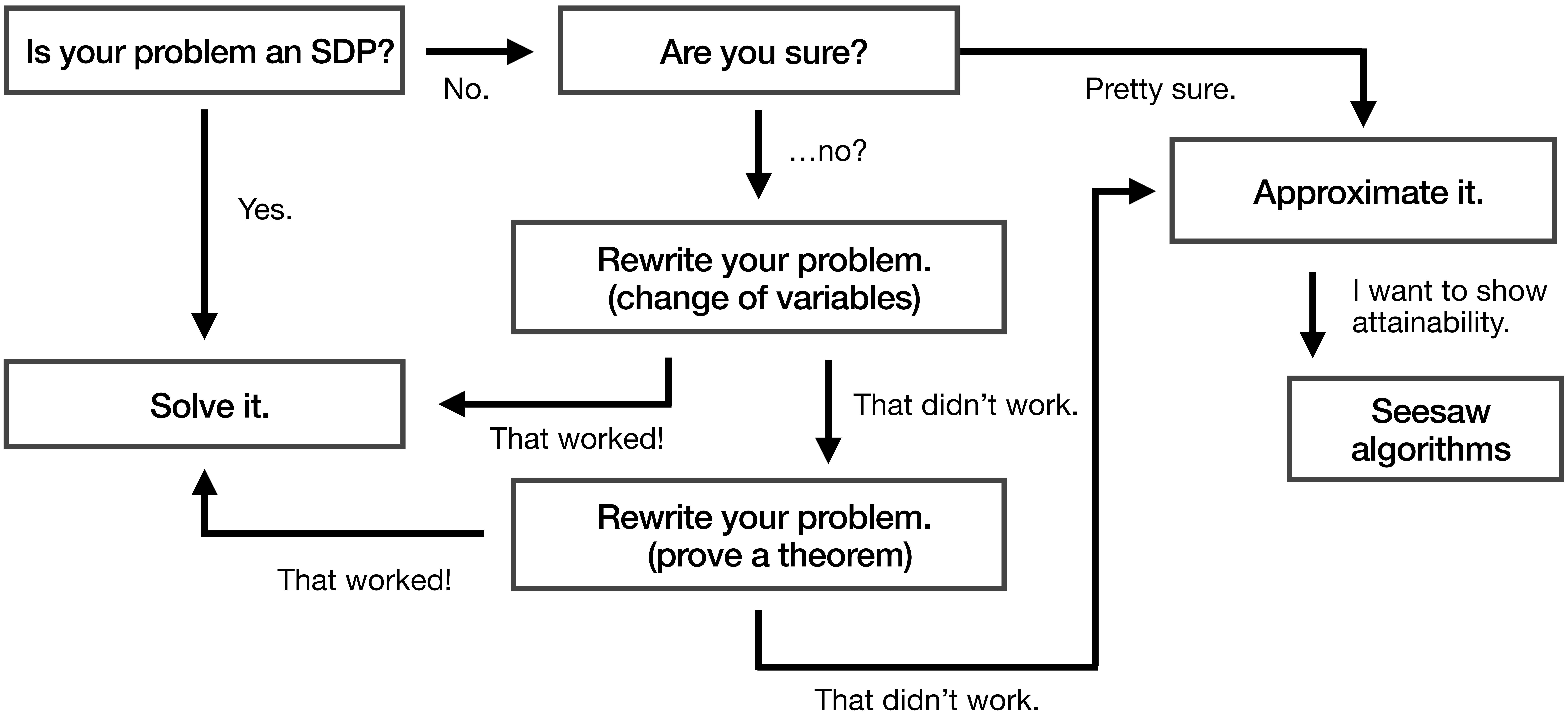

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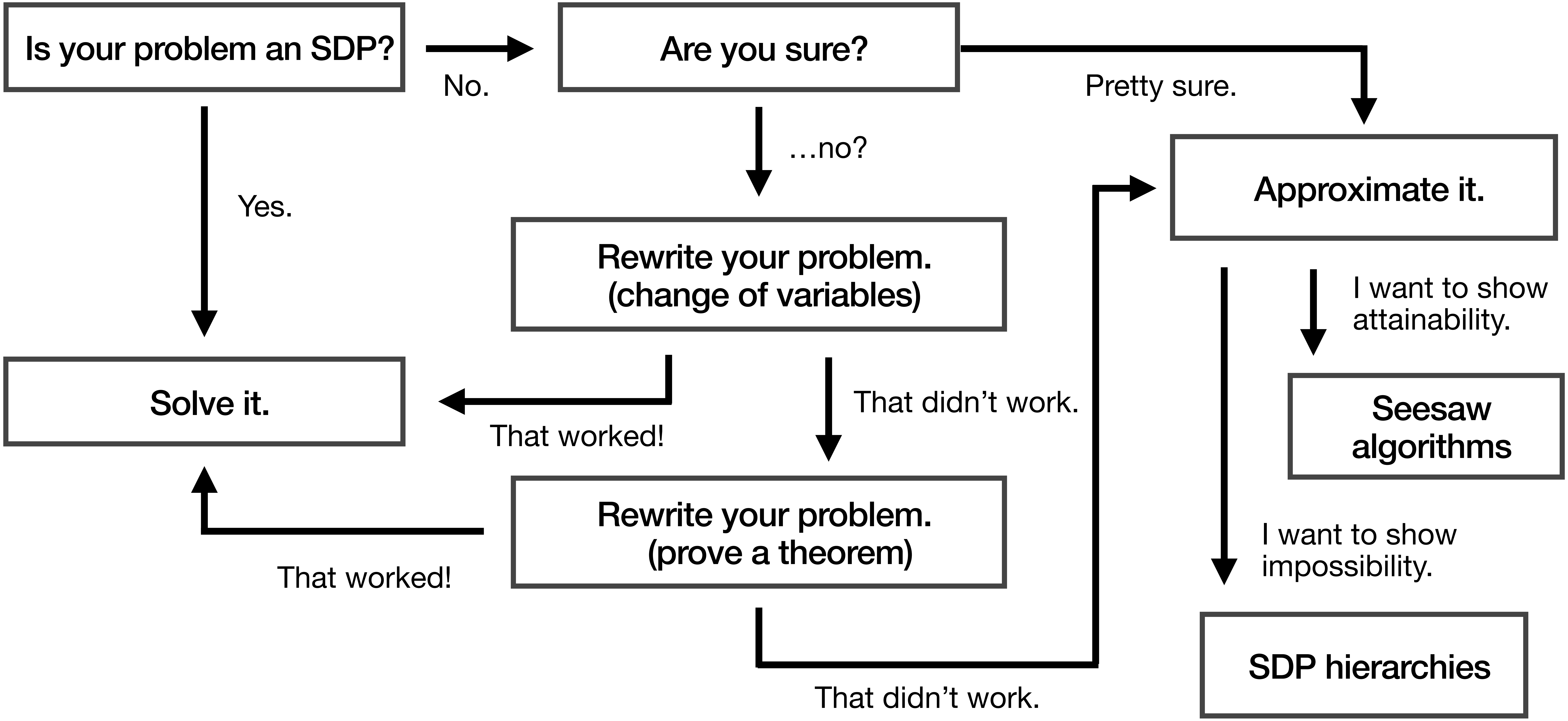

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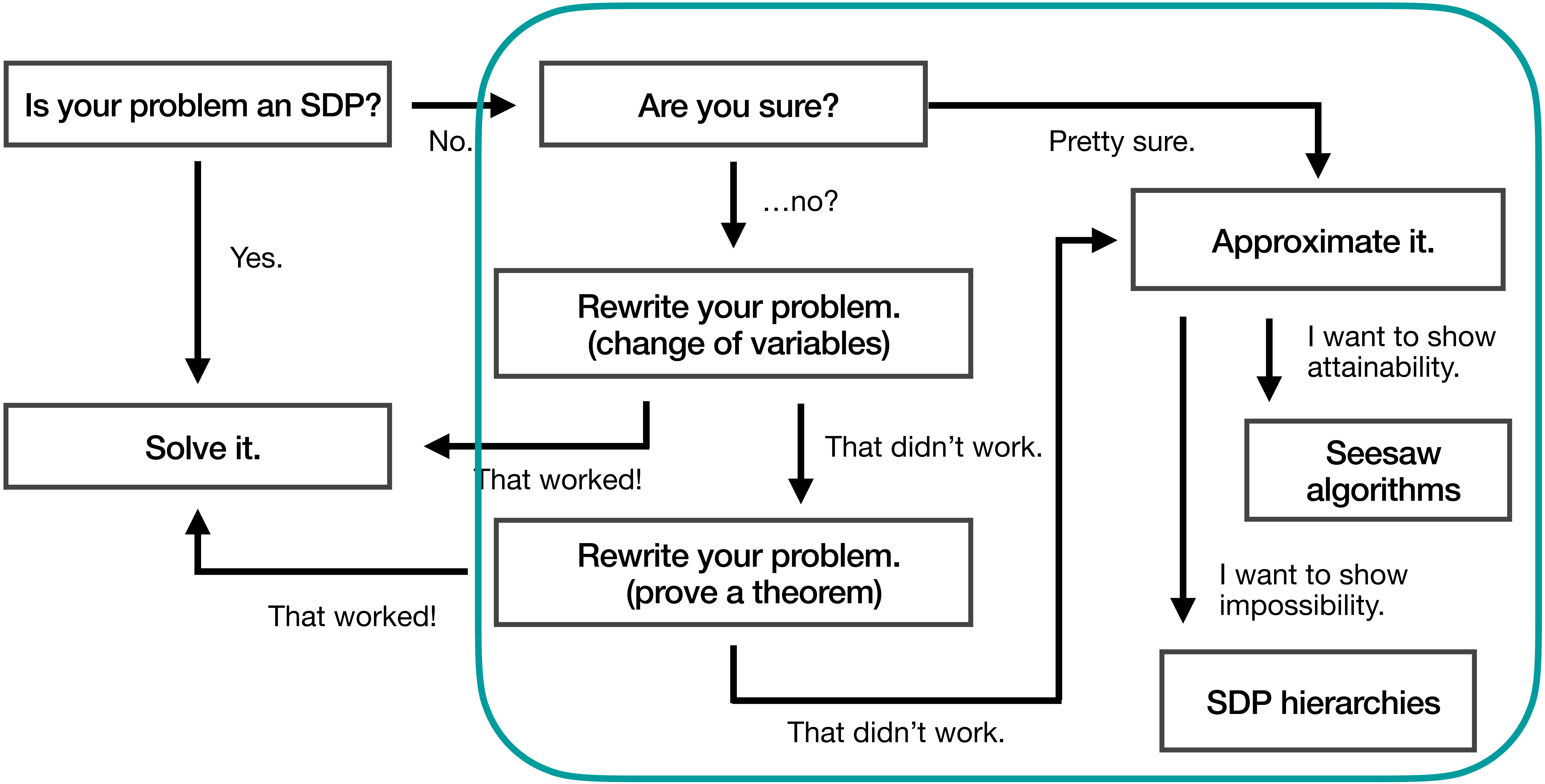


Course outline

- **I. Rewriting**

  — Variable change (easy version): Generalized noise robustness of **incompatibility**

  — External proof (advanced version): **Channel discrimination** and the diamond distance

- **II. Approximating**

  — **Attainability**

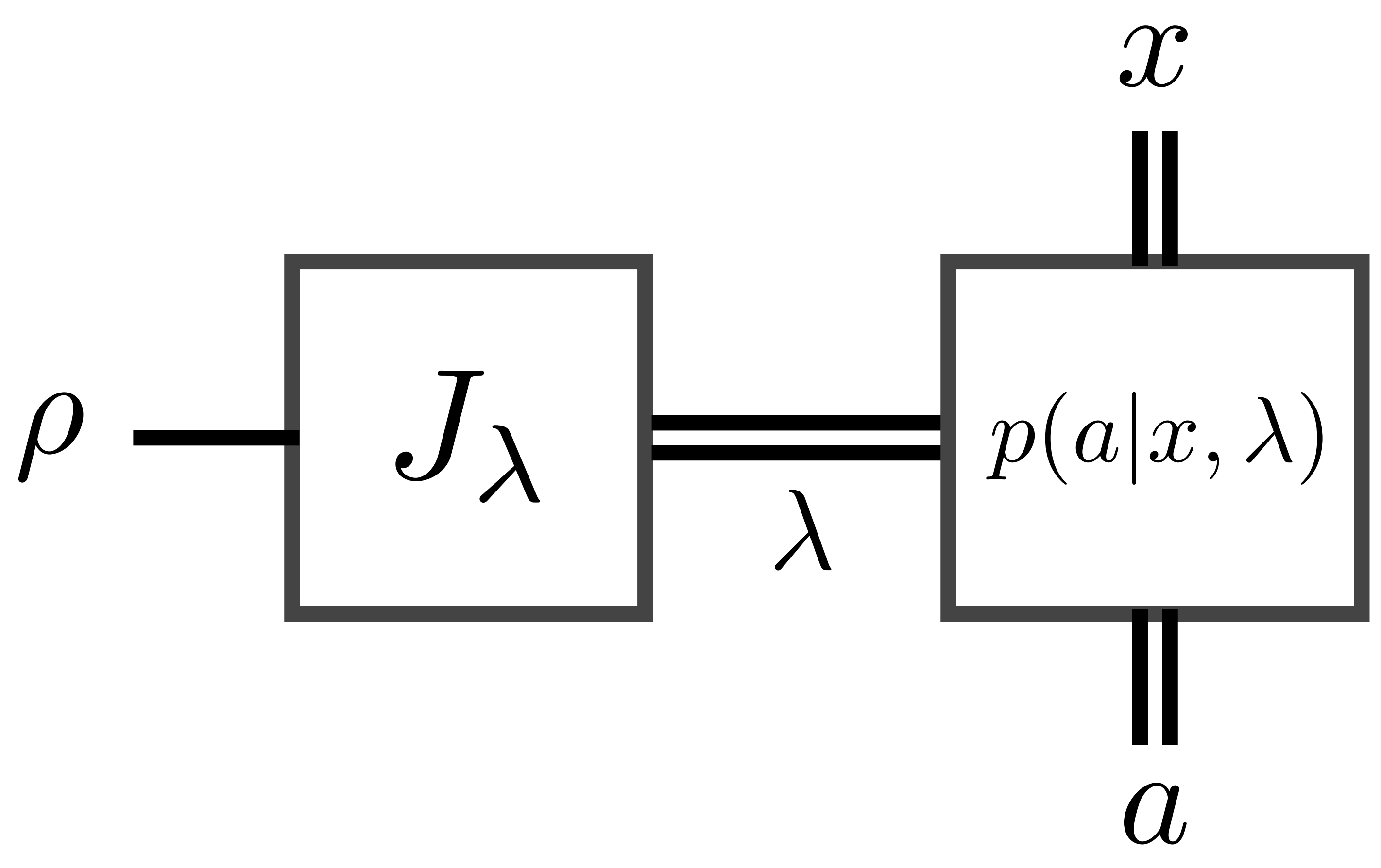
  - Seesaw algorithms (easy version): **Channel discrimination** without memory
  - Seesaw algorithms (advanced version): Most **incompatible measurements**

  — **Impossibility**

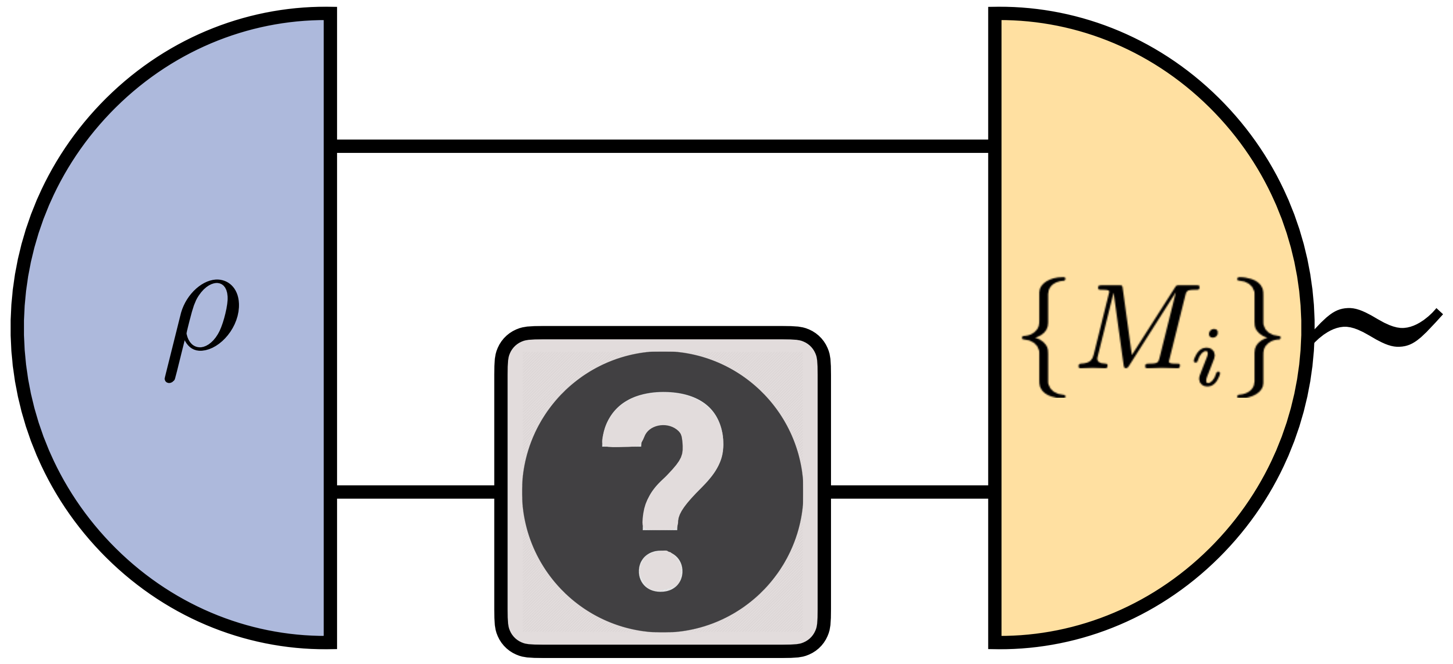
  - Outer approx. (easy version): **Channel discrimination** without memory
  - SDP hierarchy (advanced version): **Entanglement** and the k-symmetric extension hierarchy

Case studies

Measurement incompatibility



Channel discrimination



Entanglement

$$\rho_{\text{sep}} = \sum_i p_i \ \rho_i^A \otimes \rho_i^B$$

I. Rewriting

I. Rewriting your problem as an SDP

Generalized noise robustness of measurement incompatibility

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Generalized noise robustness of measurement incompatibility

Definition. A set of quantum measurements is **compatible** when it can be expressed as a single quantum measurement and classical post-processing.

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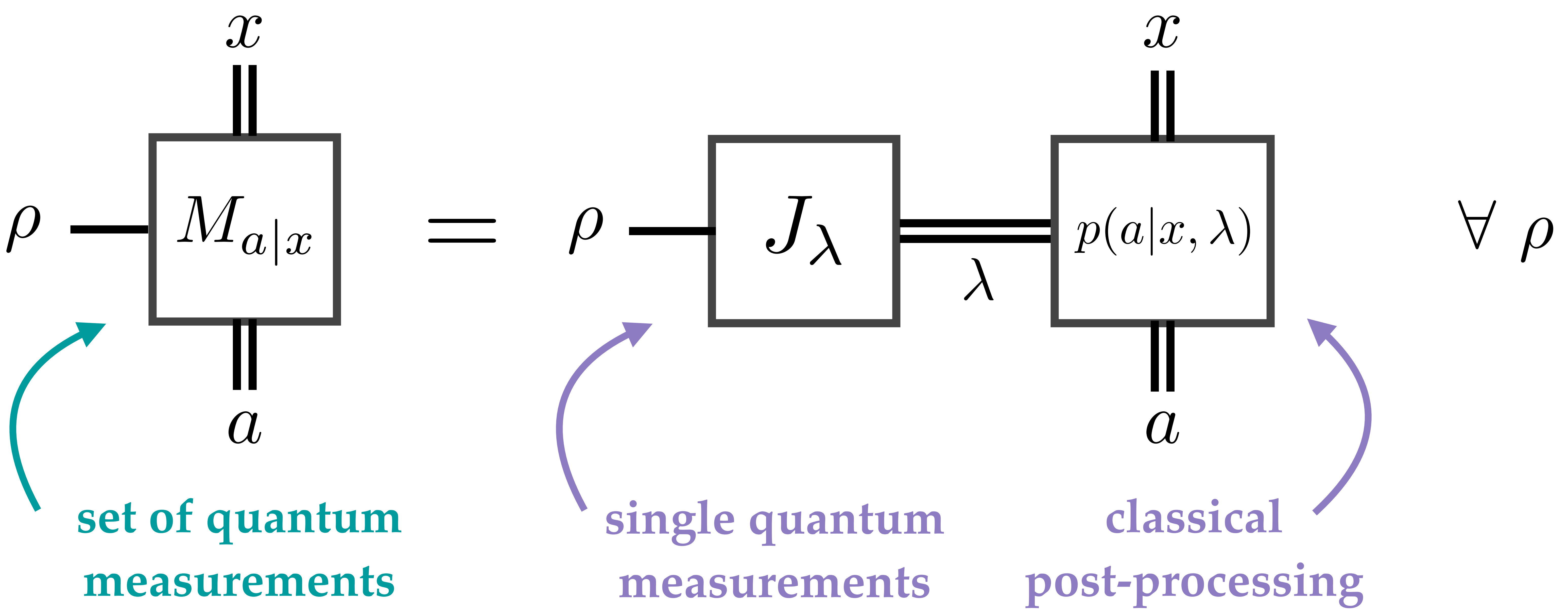

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Generalized noise robustness of measurement incompatibility

Definition. A set of quantum measurements is **compatible** when it can be expressed as a single quantum measurement and classical post-processing.

set of quantum measurements: $\{M_{a|x}\}, \quad M_{a|x} \geq 0 \ \ \forall\, a, x, \quad \sum_a M_{a|x} = \mathbb{1} \ \forall\, x$

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compatible set of quantum measurements:

$$\exists\ \{J_\lambda\}, \quad J_\lambda \geq 0 \ \ \forall\, \lambda, \quad \sum_\lambda J_\lambda = \mathbb{1}$$

$$\exists\ \{p(a|x,\lambda)\}, \quad p(a|x,\lambda) \geq 0 \ \ \forall\, a, x, \lambda, \quad \sum_a p(a|x,\lambda) = 1 \ \forall\, x, \lambda$$

$$\boxed{M_{a|x} = \sum_\lambda p(a|x,\lambda) J_\lambda \ \ \forall\, a, x}$$

I. Rewriting your problem as an SDP

Generalized noise robustness of measurement incompatibility

Question. How much "worst-case" noise can I mix into a set of measurements before it becomes compatible?

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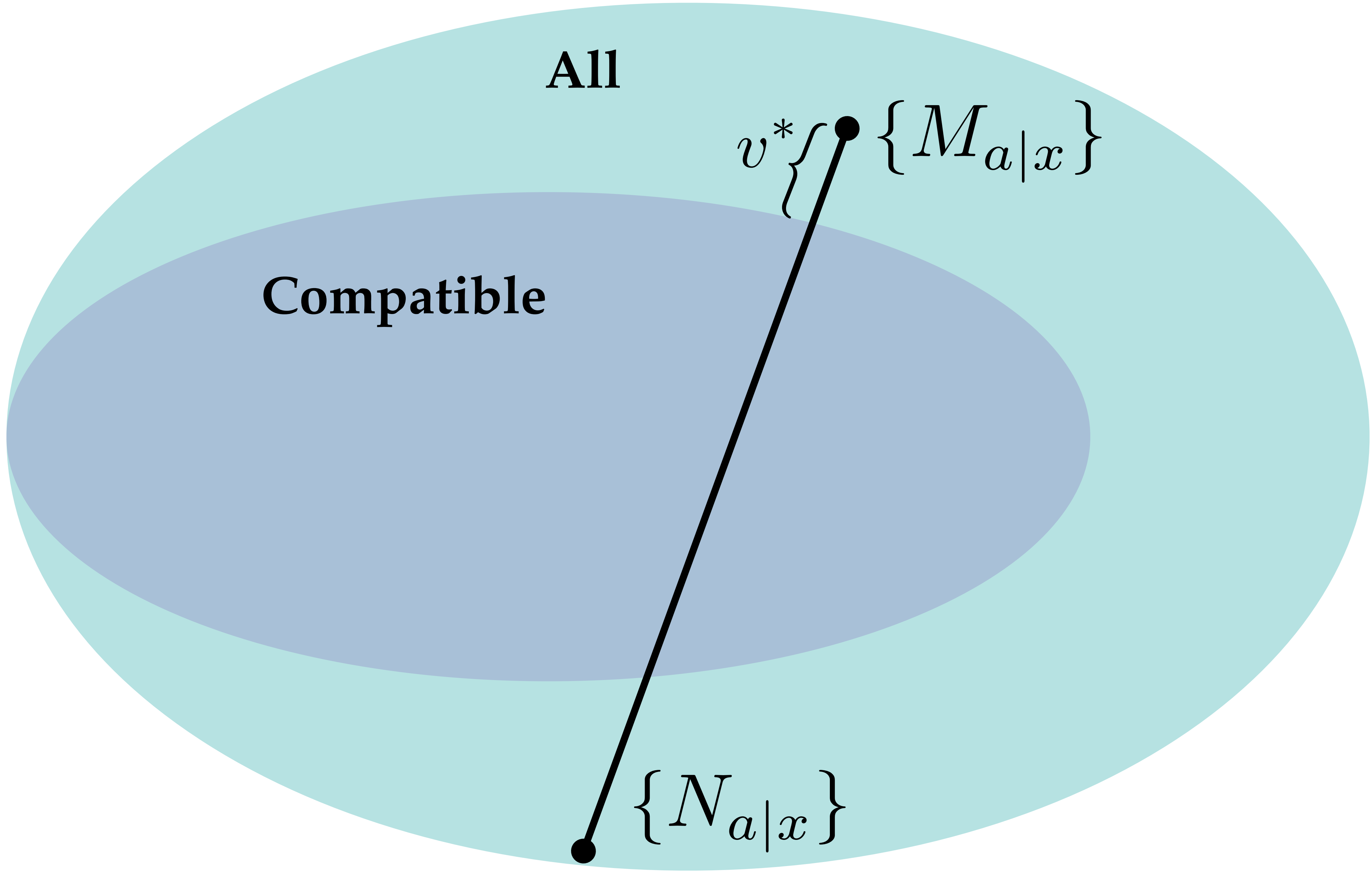

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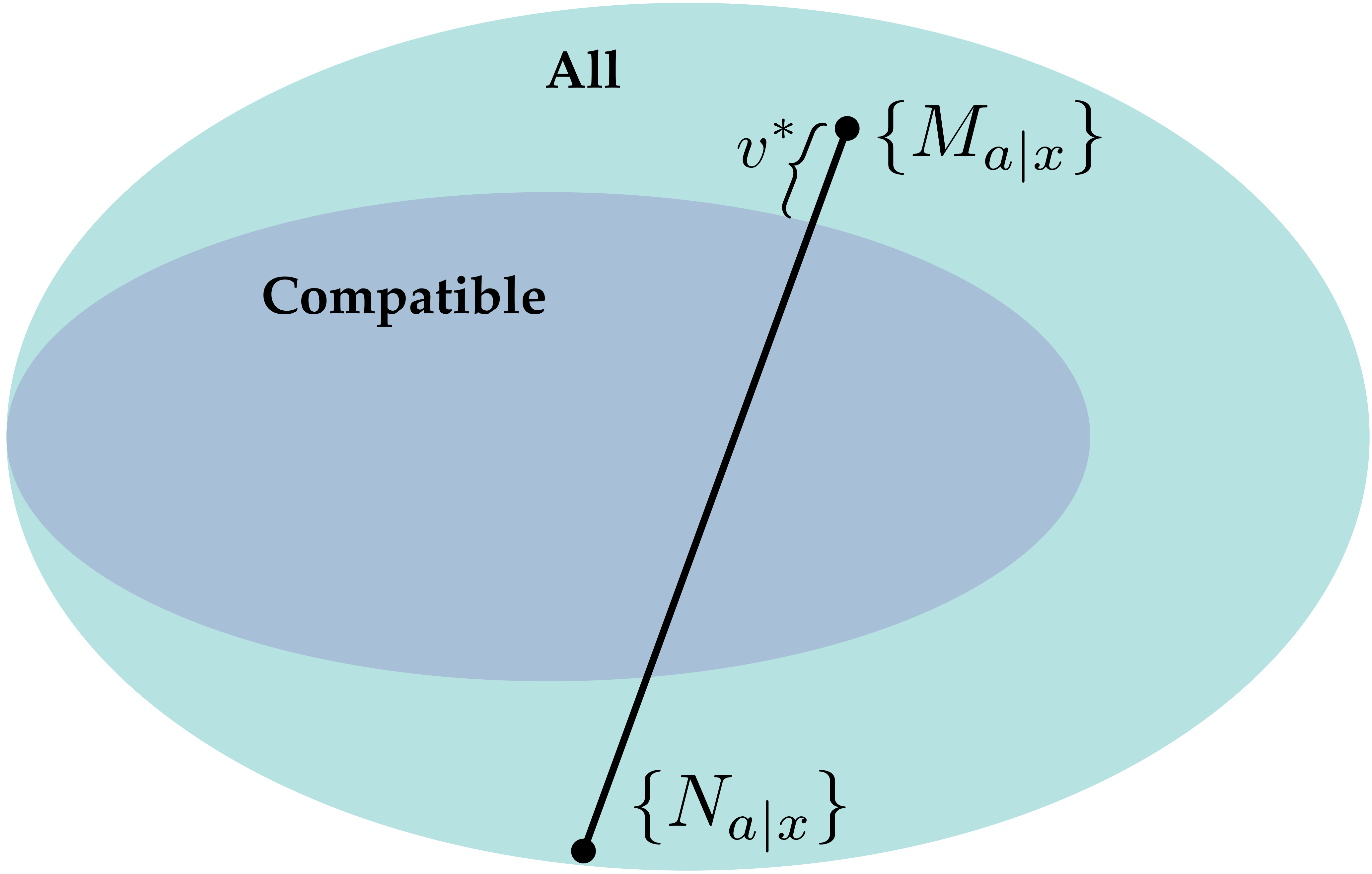

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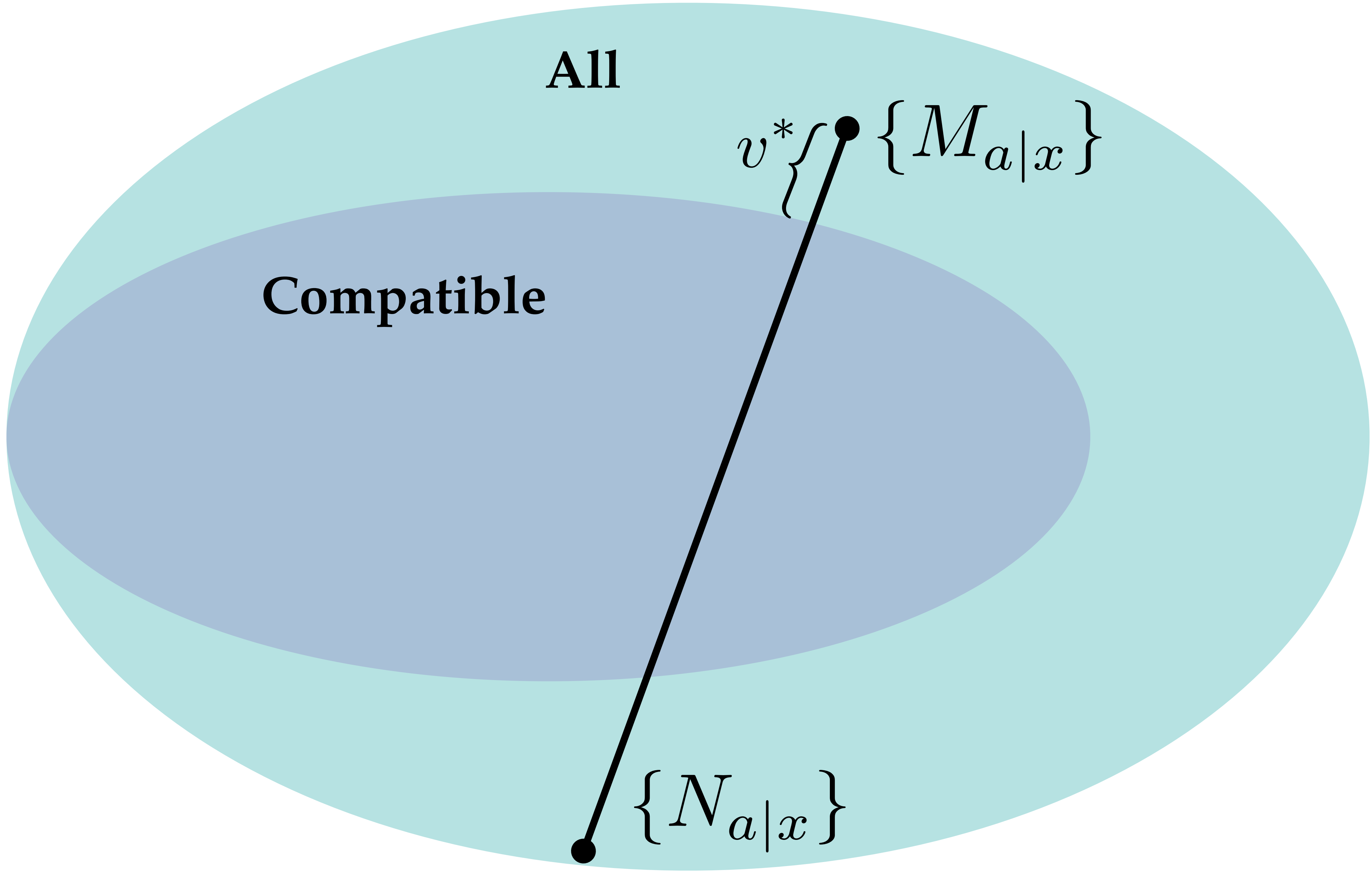

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Generalized noise robustness of measurement incompatibility

$$p(a|x,\lambda) = \sum_{\mu} \pi(\mu|\lambda) D(a|x,\mu) \longrightarrow \text{decomposition into deterministic distributions}$$

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$$\sum_{\lambda} p(a|x,\lambda) J_{\lambda} = \sum_{\lambda} \sum_{\mu} \pi(\mu|\lambda) D(a|x,\mu) J_{\lambda}$$

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Generalized noise robustness of measurement incompatibility

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Change of variable

$$J_\lambda \mapsto G_\mu = \sum_\lambda \pi(\mu|\lambda) J_\lambda$$

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$$\begin{aligned} \sum_\lambda p(a|x,\lambda) J_\lambda &= \sum_\lambda \sum_\mu \pi(\mu|\lambda) D(a|x,\mu) J_\lambda \\ &= \sum_\mu D(a|x,\mu) G_\mu \end{aligned}$$

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1) $G_\mu = \sum_\lambda \pi(\mu|\lambda) J_\lambda \geq 0,$

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$$v\, M_{a|x} + (1-v) N_{a|x} = \sum_{\lambda} p(a|x,\lambda) J_\lambda \quad \mapsto \quad v\, M_{a|x} + (1-v) N_{a|x} = \sum_{\mu} D(a|x,\mu) G_\mu$$

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Generalized noise robustness of measurement incompatibility

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$$v \mapsto \frac{1}{t+1} \qquad\qquad 0 \leq v \leq 1 \iff t \geq 0$$

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$$N_{a|x} = \frac{(t+1)\sum_\mu D(a|x,\mu) G_\mu - M_{a|x}}{t}$$

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$$N_{a|x} = \frac{(t+1)\sum_\mu D(a|x,\mu) G_\mu - M_{a|x}}{t}$$

1) $$N_{a|x} \geq 0 \iff (t+1)\sum_\mu D(a|x,\mu) G_\mu - M_{a|x} \geq 0$$

2) $$\sum_a N_{a|x} = \mathbb{1} \iff \sum_a \frac{(t+1)\sum_\mu D(a|x,\mu) G_\mu - M_{a|x}}{t} = \frac{(t+1)\mathbb{1} - \mathbb{1}}{t} = \mathbb{1} \quad \checkmark$$

I. Rewriting your problem as an SDP

Generalized noise robustness of measurement incompatibility

given $\{M_{a|x}\}$

$$t^* = \min \ t$$

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$$t \geq 0$$

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$$\Longleftrightarrow$$

Rewritten

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$\boxed{\text{SDP!}}$

$$\Longleftrightarrow$$

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I. Rewriting your problem as an SDP

Channel discrimination and the diamond distance

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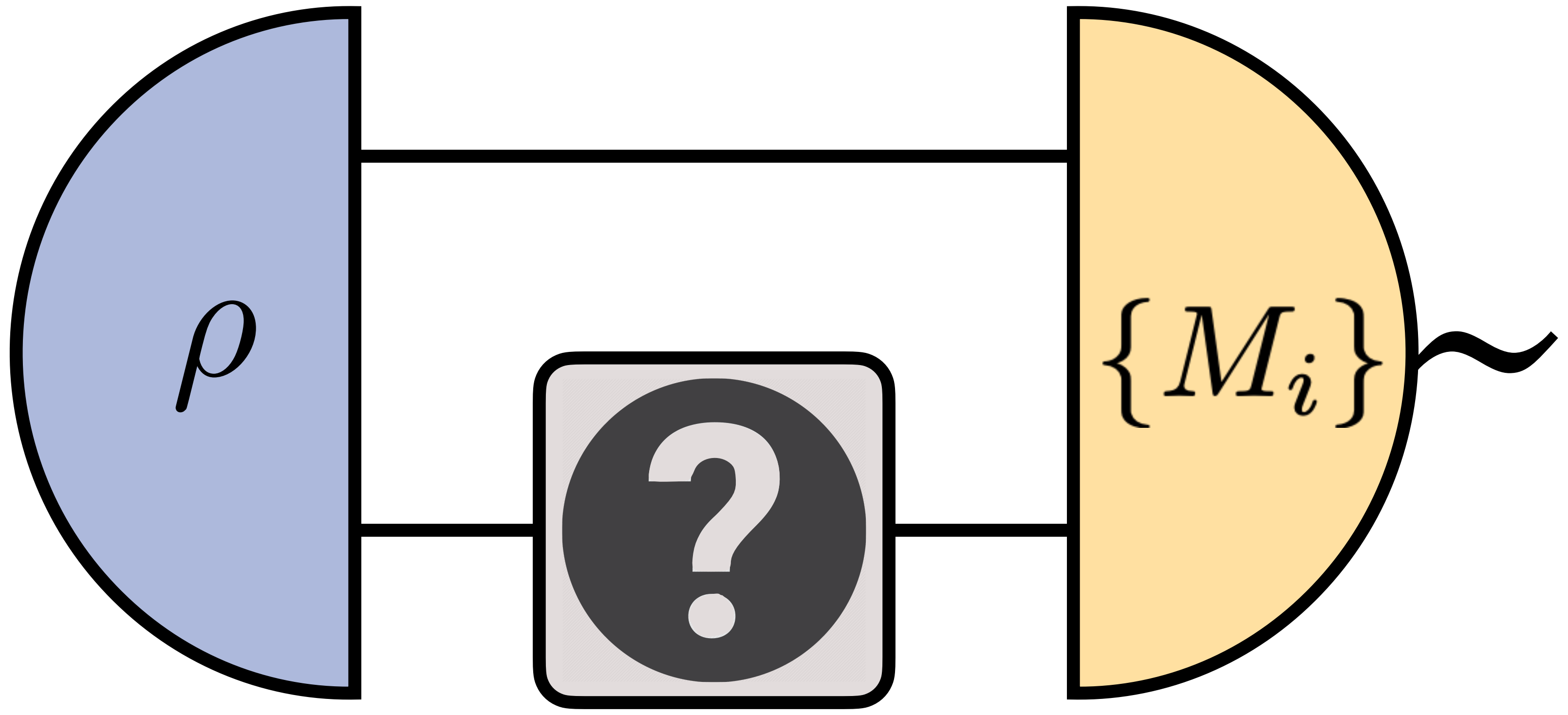
Channel discrimination and the diamond distance

$$\mathcal{E} = \{p_i, \widetilde{C}_i\}$$



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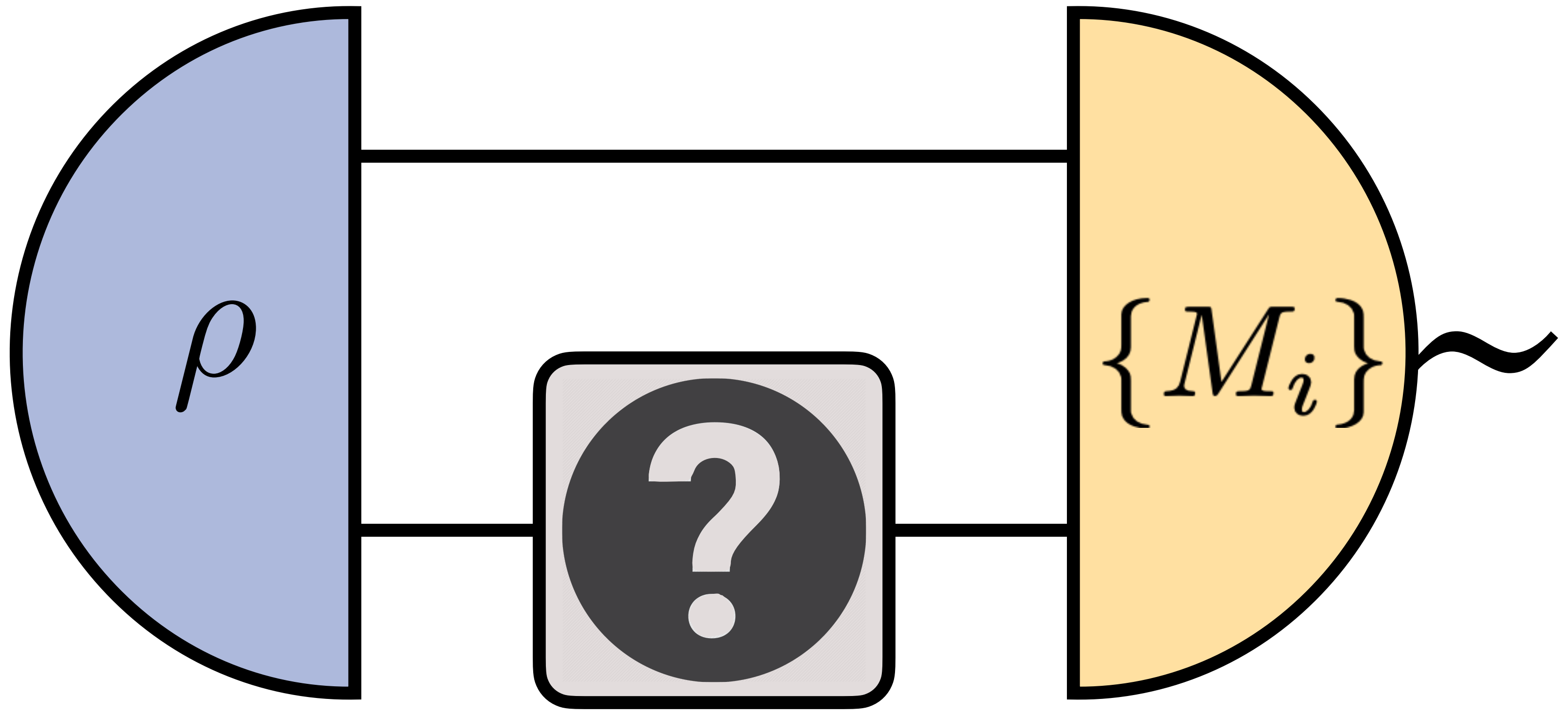

$$P = \max_{\rho, \{M_i\}} \quad \sum_i p_i \, \mathrm{tr}\left[\widetilde{C}_i \otimes \widetilde{\mathbb{1}}(\rho) \, M_i\right]$$

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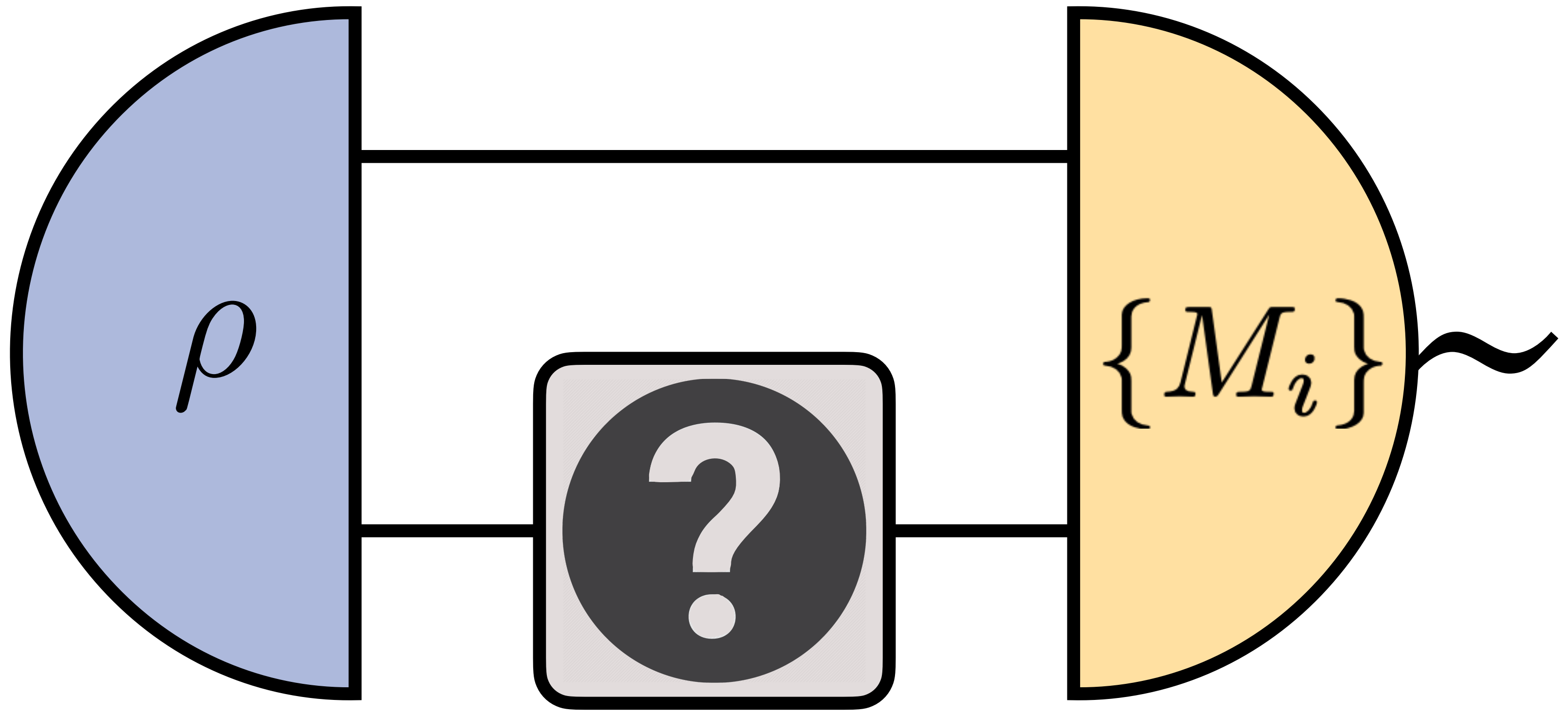

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$$\text{s.t.} \quad \rho \geq 0, \quad \operatorname{tr}(\rho) = 1$$

$$M_i \geq 0, \quad \sum_i M_i = \mathbb{1}$$

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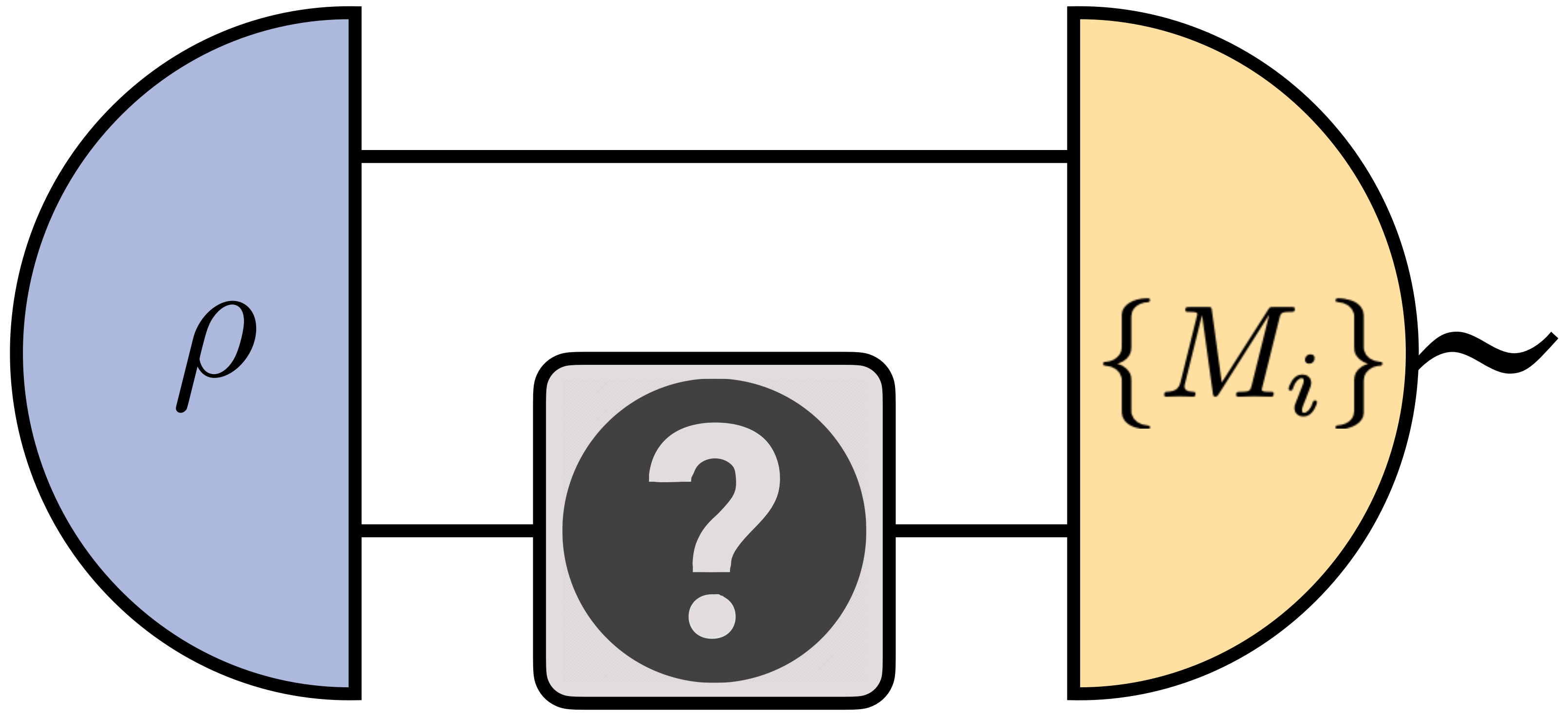
Channel discrimination and the diamond distance



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Channel discrimination and the diamond distance

Choi operator: Every linear map can be represented by a linear operator, acting on the joint input and output space of the map, defined as

$$\widetilde{C} : \mathcal{L}(\mathcal{H}_I) \to \mathcal{L}(\mathcal{H}_O) \quad \longrightarrow \text{ linear map}$$

$$C := \sum_{ij} |i\rangle\langle j| \otimes \widetilde{C}(|i\rangle\langle j|) \in \mathcal{L}(\mathcal{H}_I \otimes \mathcal{H}_O) \quad \longrightarrow \text{ linear operator}$$

Quantum channels:

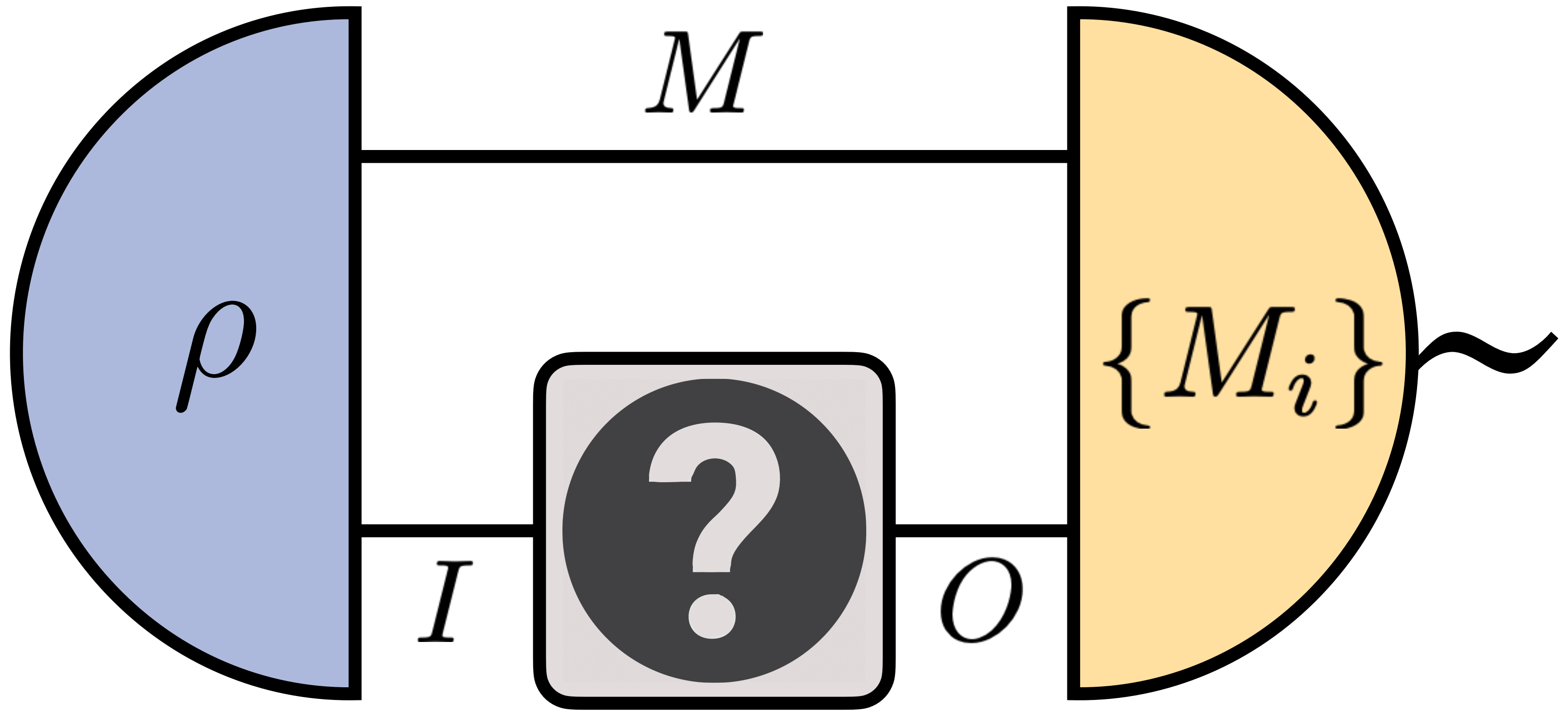
$$\widetilde{C} \in \text{CPTP} \iff C \geq 0, \quad \operatorname{tr}_O C = \mathbb{1}_I$$

Output state: Consequently, the output of a linear map can be expressed as function of the Choi operator of the map and its input according to

$$\widetilde{C}(\rho) = \rho' = \operatorname{tr}_I[C\,(\rho^T \otimes \mathbb{1}_O)]$$

I. Rewriting your problem as an SDP

Channel discrimination and the diamond distance



$$\text{given} \quad \{p_i, C_i\}$$

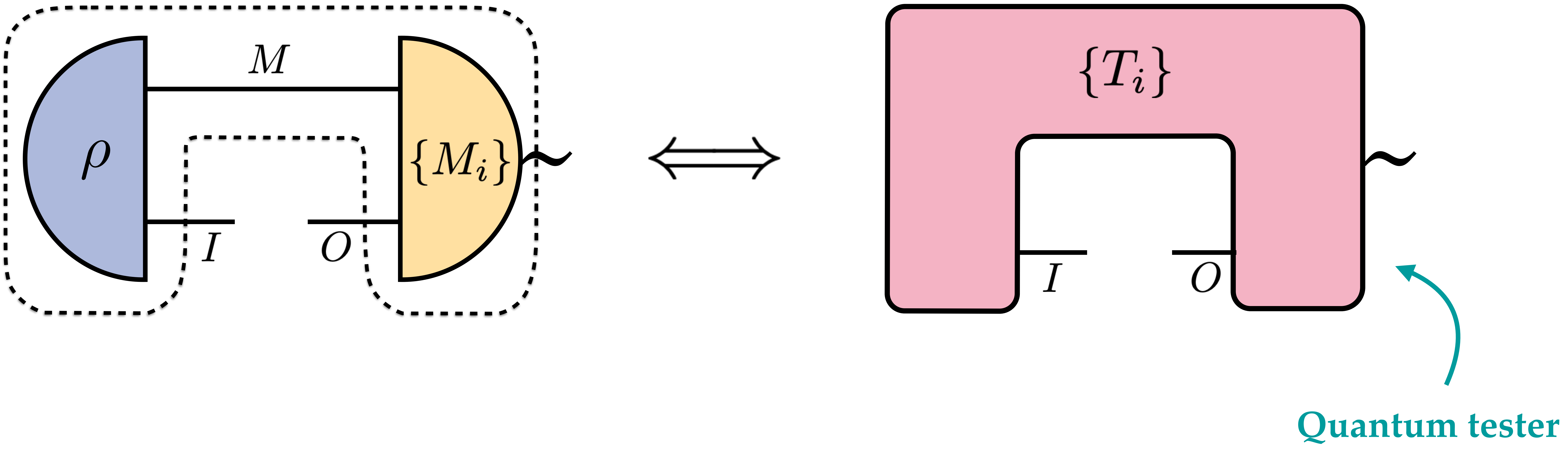
$$P = \max_{\rho,\{M_i\}} \quad \sum_i p_i \operatorname{tr}\left[(C_i \otimes \mathbb{1}_M)(\rho^{T_I} \otimes \mathbb{1}_O)(\mathbb{1}_I \otimes M_i)\right]$$

$$\text{s.t.} \quad \rho \geq 0, \quad \operatorname{tr}(\rho) = 1$$

$$M_i \geq 0, \quad \sum_i M_i = \mathbb{1}$$

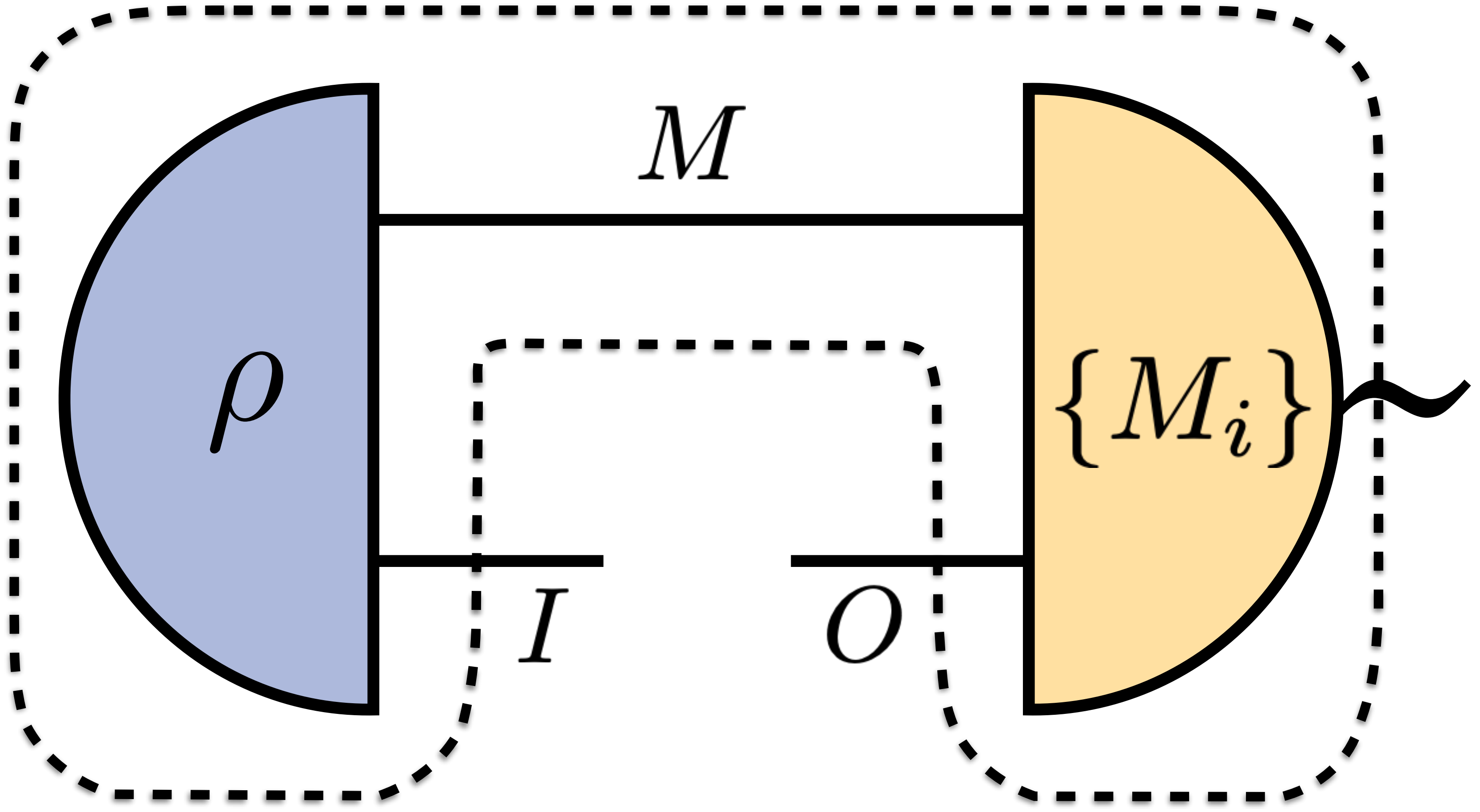
I. Rewriting your problem as an SDP

Channel discrimination and the diamond distance

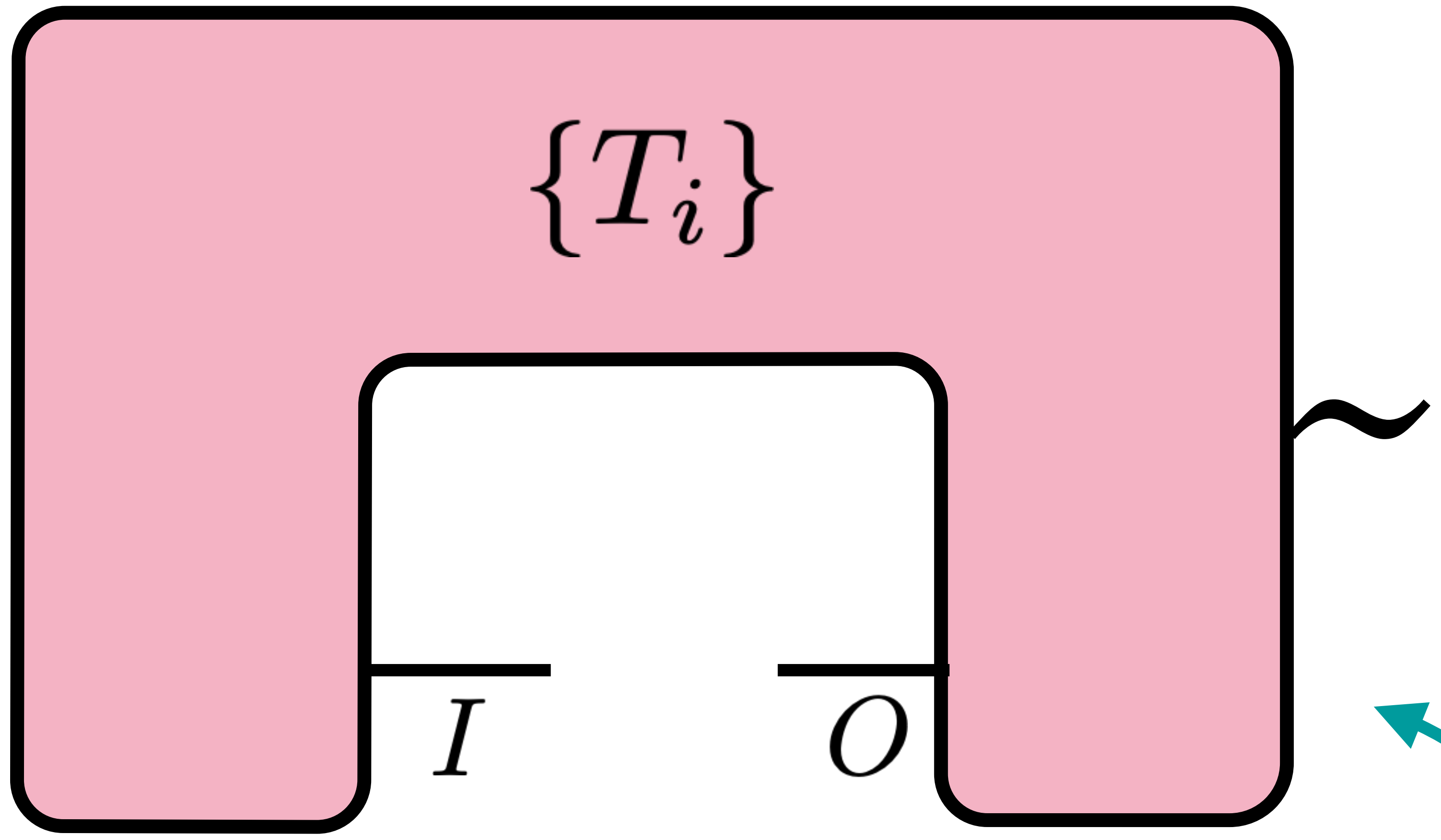


I. Rewriting your problem as an SDP

Channel discrimination and the diamond distance



$\iff$



Quantum tester

$$\rho \geq 0, \quad \mathrm{tr}(\rho) = 1$$

$$M_i \geq 0, \quad \sum_i M_i = \mathbb{1}$$

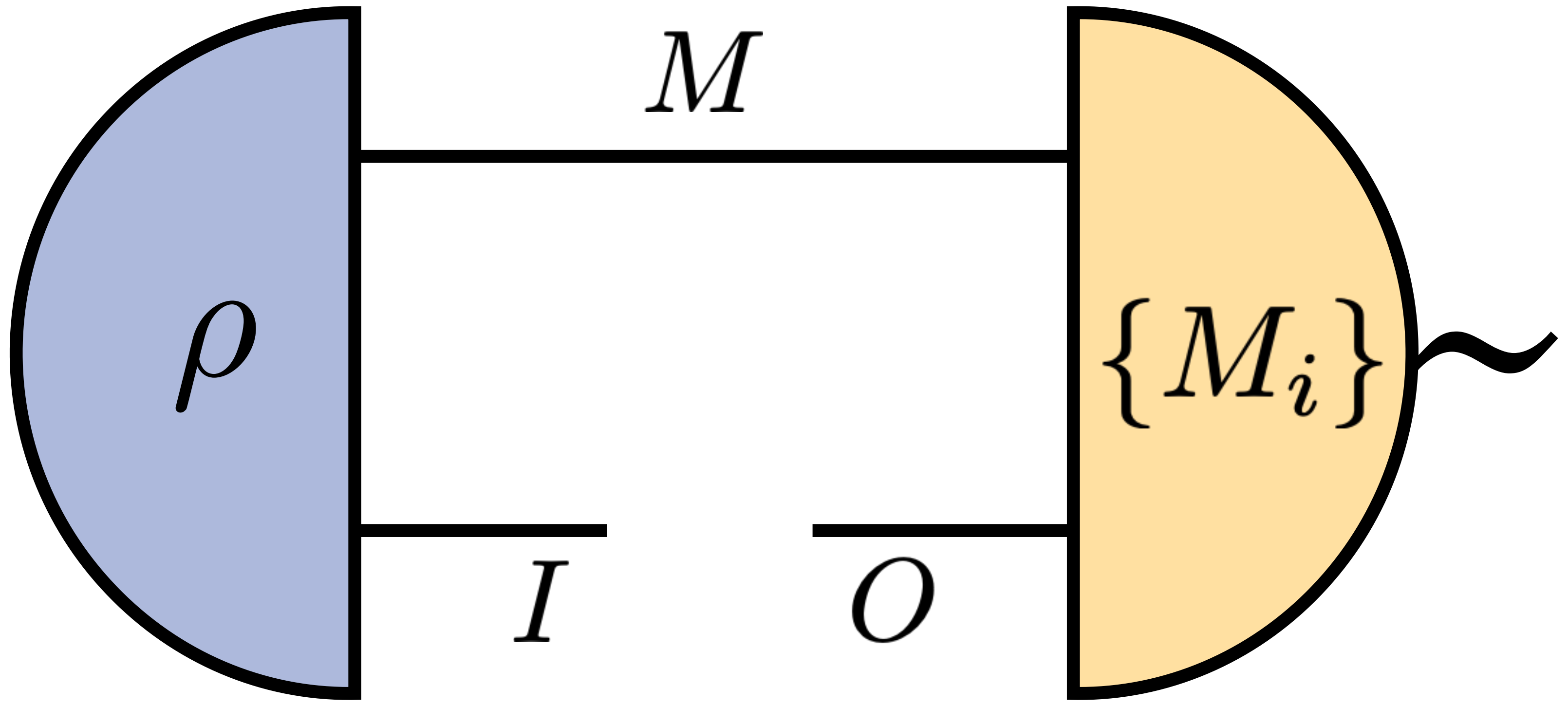
$$T_i \geq 0$$

$$\mathrm{tr}\left(\sum_i T_i\right) = d_O$$

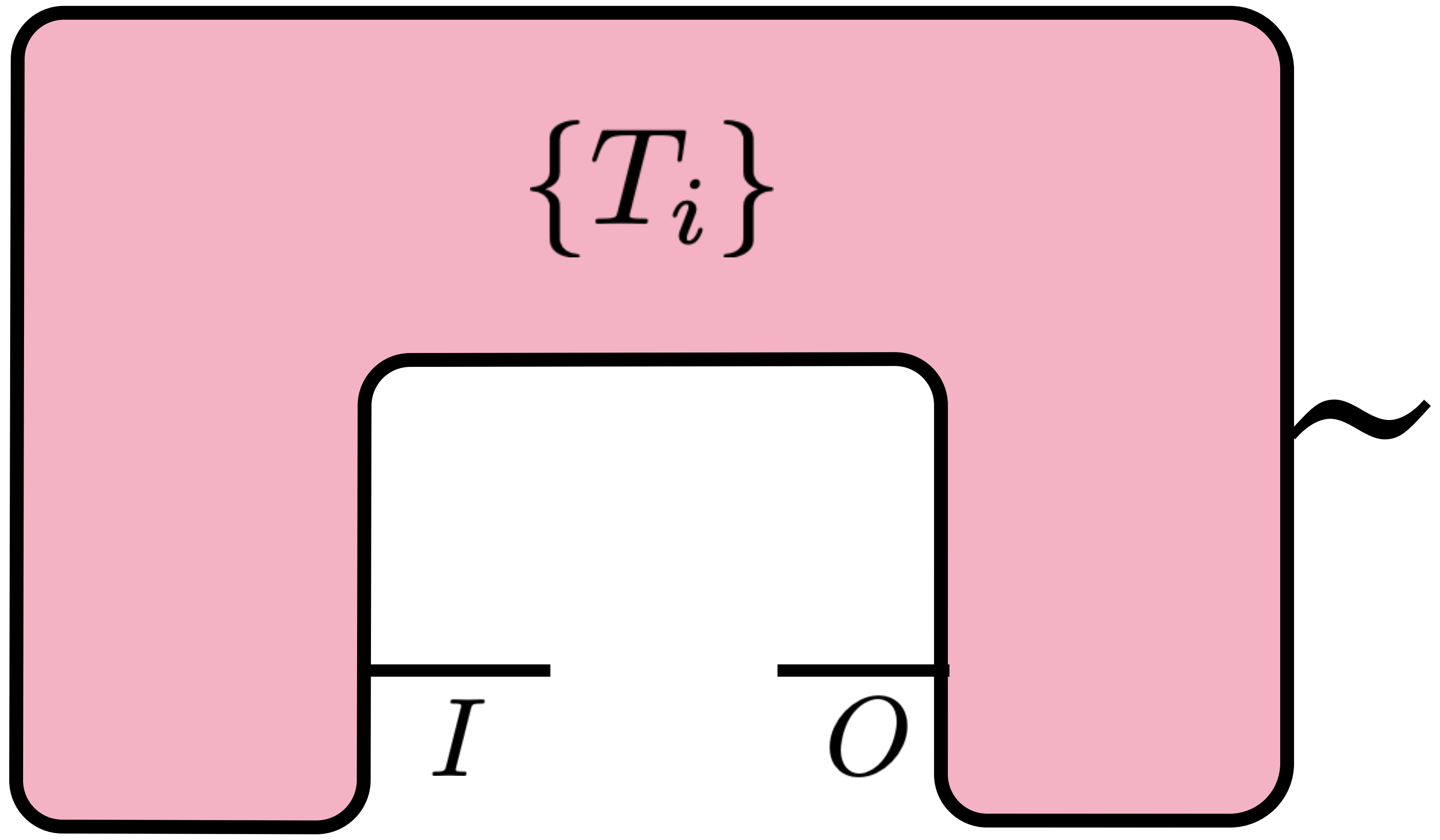
$$\sum_i T_i = \mathrm{tr}_O\left(\sum_i T_i\right) \otimes \frac{\mathbb{1}_O}{d_O}$$

I. Rewriting your problem as an SDP

Channel discrimination and the diamond distance



$\Longleftrightarrow$



$\rho, \{M_i\}$ **as a function of** $\{T_i\}$

$$\rho = \left(\mathbb{1}_I \otimes \sqrt{\sigma}^T\right) \sum_{ij} |ii\rangle\langle jj| \left(\mathbb{1}_I \otimes \sqrt{\sigma}^T\right)$$

$$M_i^T = (\sqrt{\sigma}^{-1} \otimes \mathbb{1}_O)\, T_i\, (\sqrt{\sigma}^{-1} \otimes \mathbb{1}_O)$$

$$\text{where } \sigma := \frac{\mathrm{tr}_O \sum_i T_i}{d_O}$$

$\{T_i\}$ **as a function of** $\rho, \{M_i\}$

$$T_i = \mathrm{tr}_M[(\rho^{T_M} \otimes \mathbb{1}_O)(\mathbb{1}_I \otimes M_i^T)]$$

I. Rewriting your problem as an SDP

Channel discrimination and the diamond distance

Original

$$\text{given} \quad \{p_i, \widetilde{C}_i\}$$

$$P = \max_{\rho, \{M_i\}} \quad \sum_i p_i \operatorname{tr}\left[\widetilde{C}_i \otimes \widetilde{\mathbb{1}}(\rho)\, M_i\right]$$

$$\text{s.t.} \quad \rho \geq 0, \quad \operatorname{tr}(\rho) = 1$$

$$M_i \geq 0, \quad \sum_i M_i = \mathbb{1}$$

$$\Longleftrightarrow$$

Rewritten

$$\text{given} \quad \{p_i, C_i\}$$

$$P = \max_{\{T_i\}} \quad \sum_i p_i \operatorname{tr}\left(C_i^T\, T_i\right)$$

$$\text{s.t.} \quad T_i \geq 0$$

$$\operatorname{tr}\left(\sum_i T_i\right) = d_O$$

$$\sum_i T_i = \operatorname{tr}_O\left(\sum_i T_i\right) \otimes \frac{\mathbb{1}_O}{d_O}$$

$\boxed{\textbf{SDP!}}$

I. Rewriting your problem as an SDP

Channel discrimination and the diamond distance

Diamond distance: The diamond distance between two quantum channels is related to the probability of successfully discriminating this pair of channels with a uniform prior.

$$\left\|\widetilde{C}_1 - \widetilde{C}_2\right\|_\diamond := \max_\rho \left\|\widetilde{C}_1 \otimes \widetilde{\mathbb{1}}(\rho) - \widetilde{C}_2 \otimes \widetilde{\mathbb{1}}(\rho)\right\|_1$$

I. Rewriting your problem as an SDP

Channel discrimination and the diamond distance

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$$P = \frac{1}{2} + \frac{1}{4}\left\|\widetilde{C}_1 - \widetilde{C}_2\right\|_\diamond$$

I. Rewriting your problem as an SDP

Channel discrimination and the diamond distance

Diamond distance: The diamond distance between two quantum channels is related to the probability of successfully discriminating this pair of channels with a uniform prior.

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$$P = \frac{1}{2} + \frac{1}{4}\left\|\widetilde{C}_1 - \widetilde{C}_2\right\|_\diamond$$

$\boxed{\text{SDP!}}$

I. Rewriting

MAIN MESSAGE:

Not every SDP is created in an SDP form.

Properties of the specific problem and changes of variables are useful tools to recognize that your problem is an SDP.

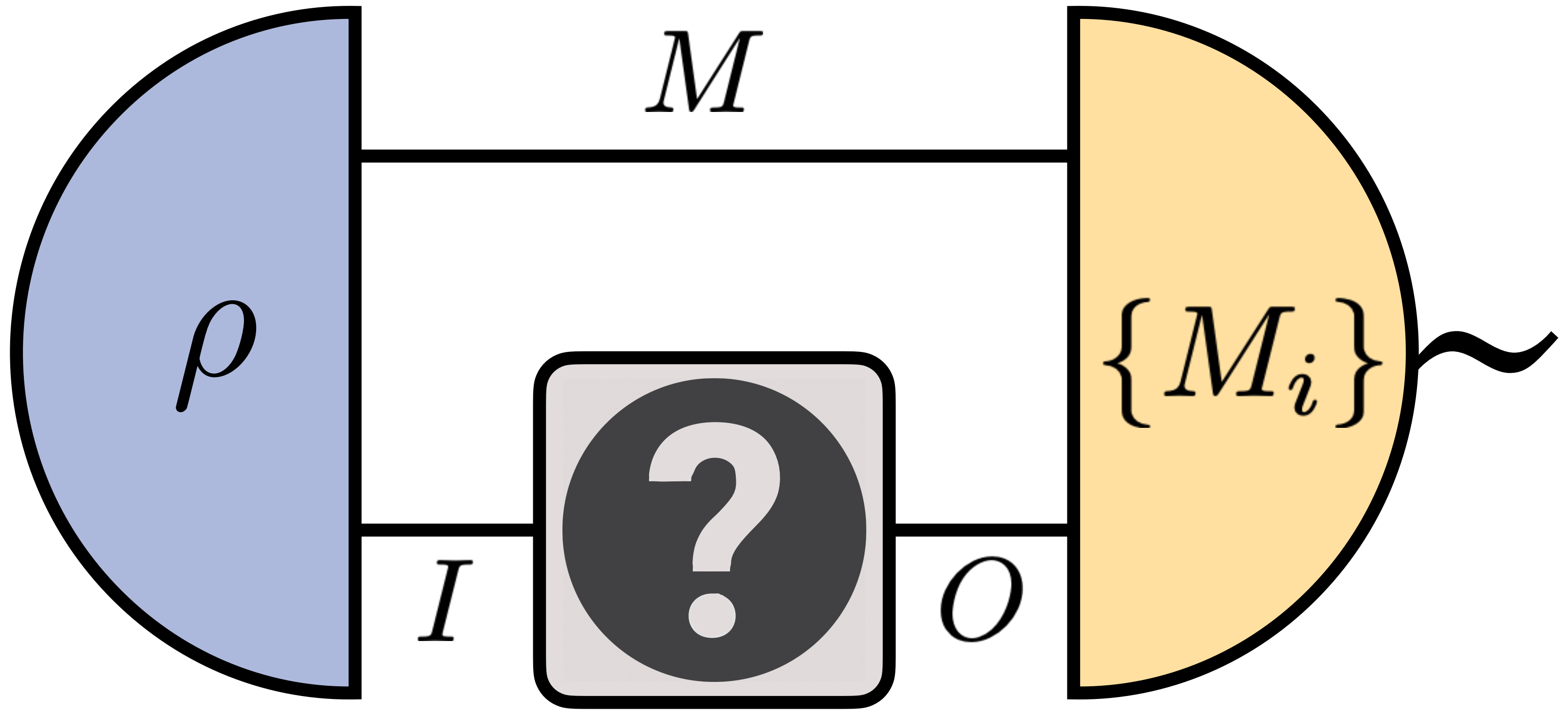
I. Rewriting

OTHER SDP REWRITING APPLICATIONS

II. Approximating: Attainability

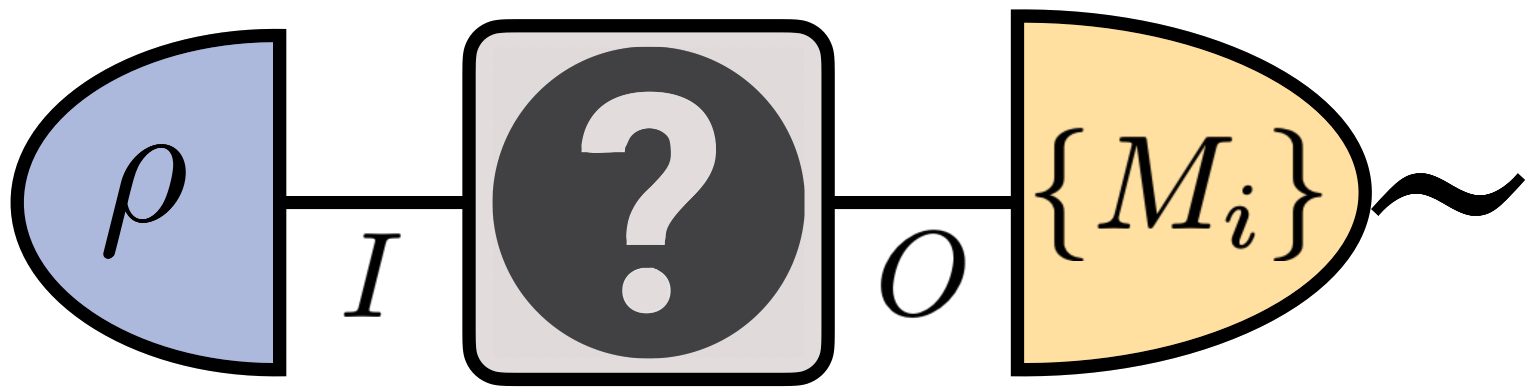
II. Approximation: Attainability

Channel discrimination without memory



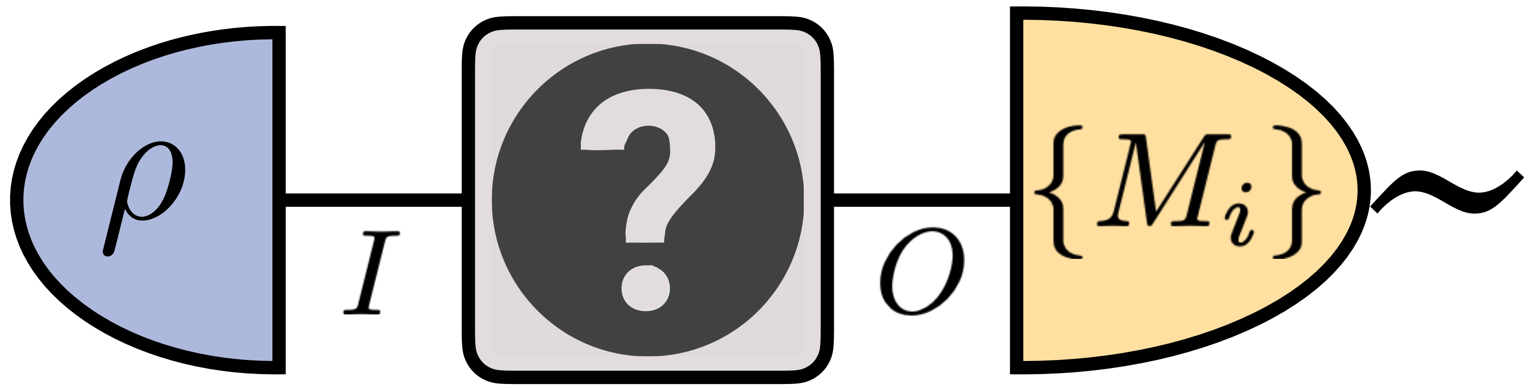
II. Approximation: Attainability

Channel discrimination without memory



II. Approximation: Attainability

Channel discrimination without memory



$$\text{given} \quad \{p_i, \widetilde{C}_i\}$$

$$P_{\mathrm{ML}} = \max_{\rho,\{M_i\}} \quad \sum_i p_i \operatorname{tr}\left[\widetilde{C}_i(\rho)\, M_i\right]$$

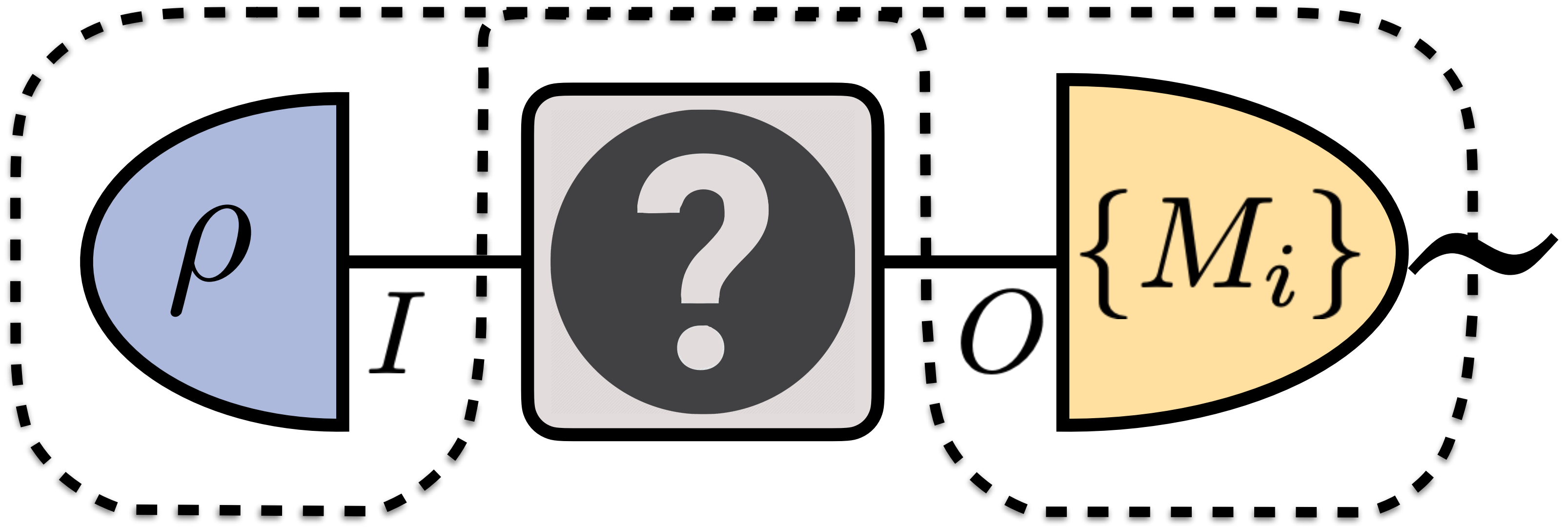
memoryless

$$\text{s.t.} \quad \rho \geq 0, \quad \operatorname{tr}(\rho) = 1$$

$$M_i \geq 0, \quad \sum_i M_i = \mathbb{1}$$

II. Approximation: Attainability

Channel discrimination without memory



$$T_i = \rho \otimes M_i^T$$

$$\text{given} \quad \{p_i, C_i\}$$

$$P_{\mathrm{ML}} = \max_{\{T_i\}} \sum_i p_i \operatorname{tr}\left[C_i^T(\rho \otimes M_i^T)\right]$$

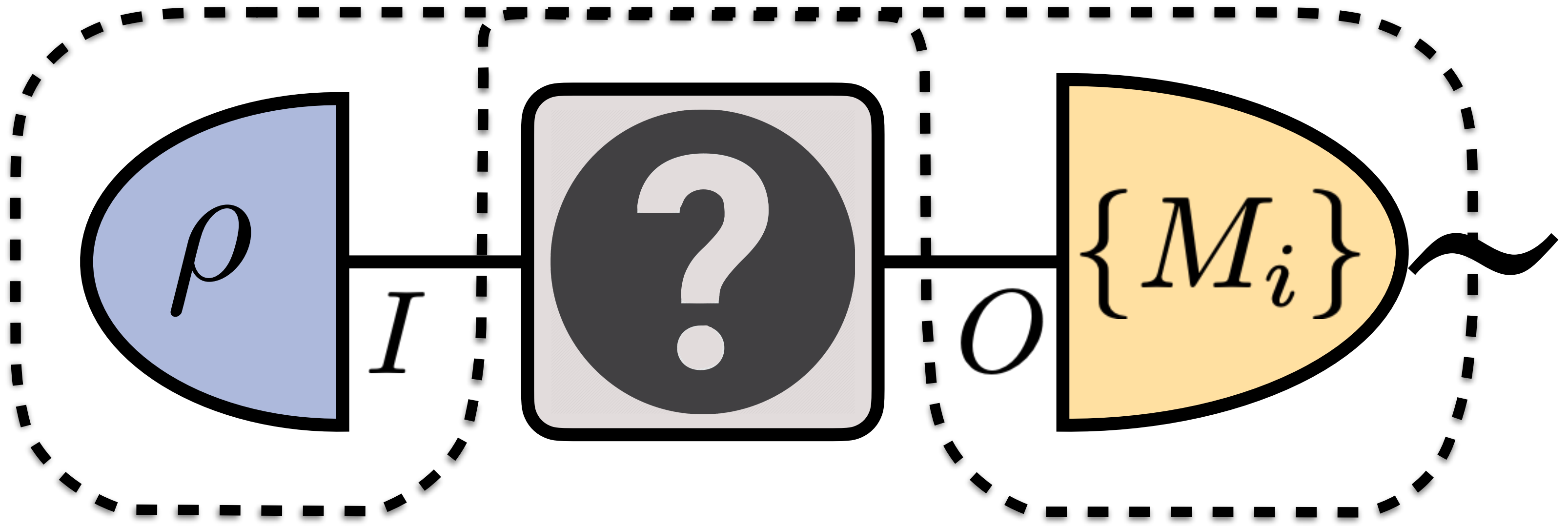
$$\text{s.t.} \quad T_i = \rho \otimes M_i^T$$

$$\rho \geq 0, \quad \operatorname{tr}(\rho) = 1$$

$$M_i \geq 0, \quad \sum_i M_i = \mathbb{1}$$

II. Approximation: Attainability

Channel discrimination without memory



$$T_i = \rho \otimes M_i^T$$

no known
SDP characterization

given $\quad \{p_i, C_i\}$

$$P_{\mathrm{ML}} = \max_{\rho} \max_{\{T_i\}} \sum_i p_i \operatorname{tr}\left[C_i^T (\rho \otimes M_i^T)\right]$$

$$\text{s.t.} \quad \rho \geq 0, \quad \operatorname{tr}(\rho) = 1$$

$$M_i \geq 0, \quad \sum_i M_i = \mathbb{1}$$

II. Approximation: Attainability

Channel discrimination without memory

Seesaw algorithm

II. Approximation: Attainability

Channel discrimination without memory

Seesaw algorithm

$$\text{given} \quad \{p_i, C_i\}, \rho$$

$$P_1 = \max_{\{M_i\}} \quad \sum_i p_i \operatorname{tr}\left(C_i^T T_i\right)$$

$$\text{s.t.} \quad T_i = \rho \otimes M_i^T$$

$$M_i \geq 0, \quad \sum_i M_i = \mathbb{1}$$

SDP 1

II. Approximation: Attainability

Channel discrimination without memory

Seesaw algorithm

given $\{p_i, C_i\}, \rho$

$$P_1 = \max_{\{M_i\}} \sum_i p_i \, \mathrm{tr}\left(C_i^T \, T_i\right)$$

$$\text{s.t.} \quad T_i = \rho \otimes M_i^T$$

$$M_i \geq 0, \quad \sum_i M_i = \mathbb{1}$$

SDP 1

II. Approximation: Attainability

Channel discrimination without memory

Seesaw algorithm

$$\text{given} \quad \{p_i, C_i\}, \rho$$

$$P_1 = \max_{\{M_i\}} \quad \sum_i p_i \operatorname{tr}\left(C_i^T T_i\right)$$

$$\text{s.t.} \quad T_i = \rho \otimes M_i^T$$

$$M_i \geq 0, \quad \sum_i M_i = \mathbb{1}$$

SDP 1

$$\text{given} \quad \{p_i, C_i\}, \{M_i\}$$

$$P_2 = \max_{\rho} \quad \sum_i p_i \operatorname{tr}\left(C_i^T T_i\right)$$

$$\text{s.t.} \quad T_i = \rho \otimes M_i^T$$

$$\rho \geq 0, \quad \operatorname{tr}(\rho) = 1$$

SDP 2

II. Approximation: Attainability

Channel discrimination without memory

Seesaw algorithm

given $\{p_i, C_i\}, \rho$

$$P_1 = \max_{\{M_i\}} \sum_i p_i \operatorname{tr}\left(C_i^T T_i\right)$$

s.t. $T_i = \rho \otimes M_i^T$

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SDP 1

given $\{p_i, C_i\}, \{M_i\}$

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SDP 2

II. Approximation: Attainability

Channel discrimination without memory

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SDP 2

II. Approximation: Attainability

Channel discrimination without memory

Algorithm 1 Seesaw SDP algorithm

Initialize ρ

$\epsilon \leftarrow 10^{-5}$

$\Delta \leftarrow \infty$

while $\Delta > \varepsilon$ **do**

    $(P_1, \{M_i\}) \leftarrow \text{SDP 1}(\{p_i, C_i\}, \rho)$

    $(P_2, \rho) \leftarrow \text{SDP 2}(\{p_i, C_i\}, \{M_i\})$

    $\Delta \leftarrow |P_1 - P_2|$

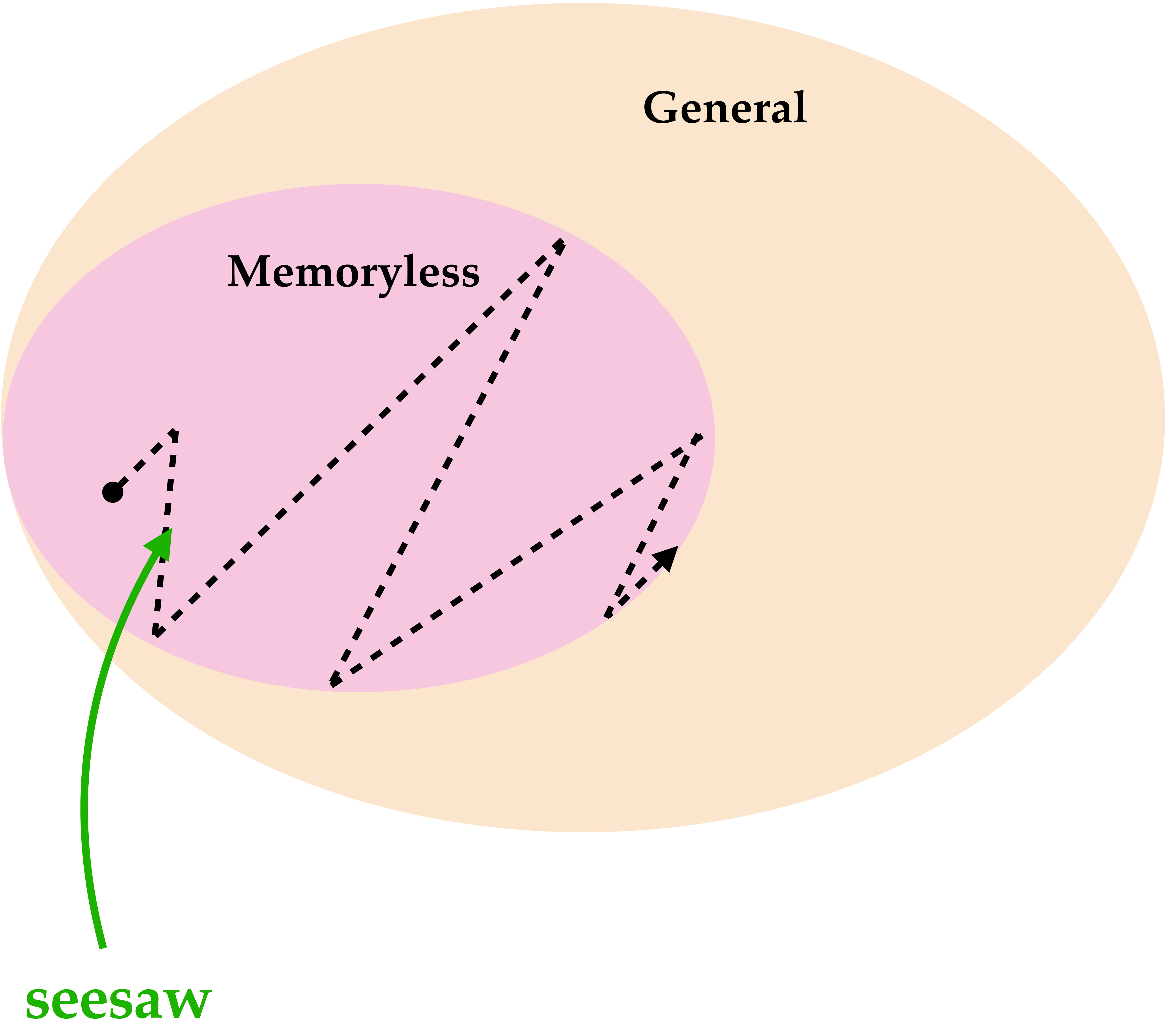
end while

$P_{\text{seesaw}} \leftarrow P_2$

return $P, \rho, \{M_i\}$

II. Approximation: Attainability

Channel discrimination without memory



Algorithm 1 Seesaw SDP algorithm

Initialize ρ

$\epsilon \leftarrow 10^{-5}$

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while $\Delta > \varepsilon$ **do**

    $(P_1, \{M_i\}) \leftarrow \text{SDP } 1(\{p_i, C_i\}, \rho)$

    $(P_2, \rho) \leftarrow \text{SDP } 2(\{p_i, C_i\}, \{M_i\})$

    $\Delta \leftarrow |P_1 - P_2|$

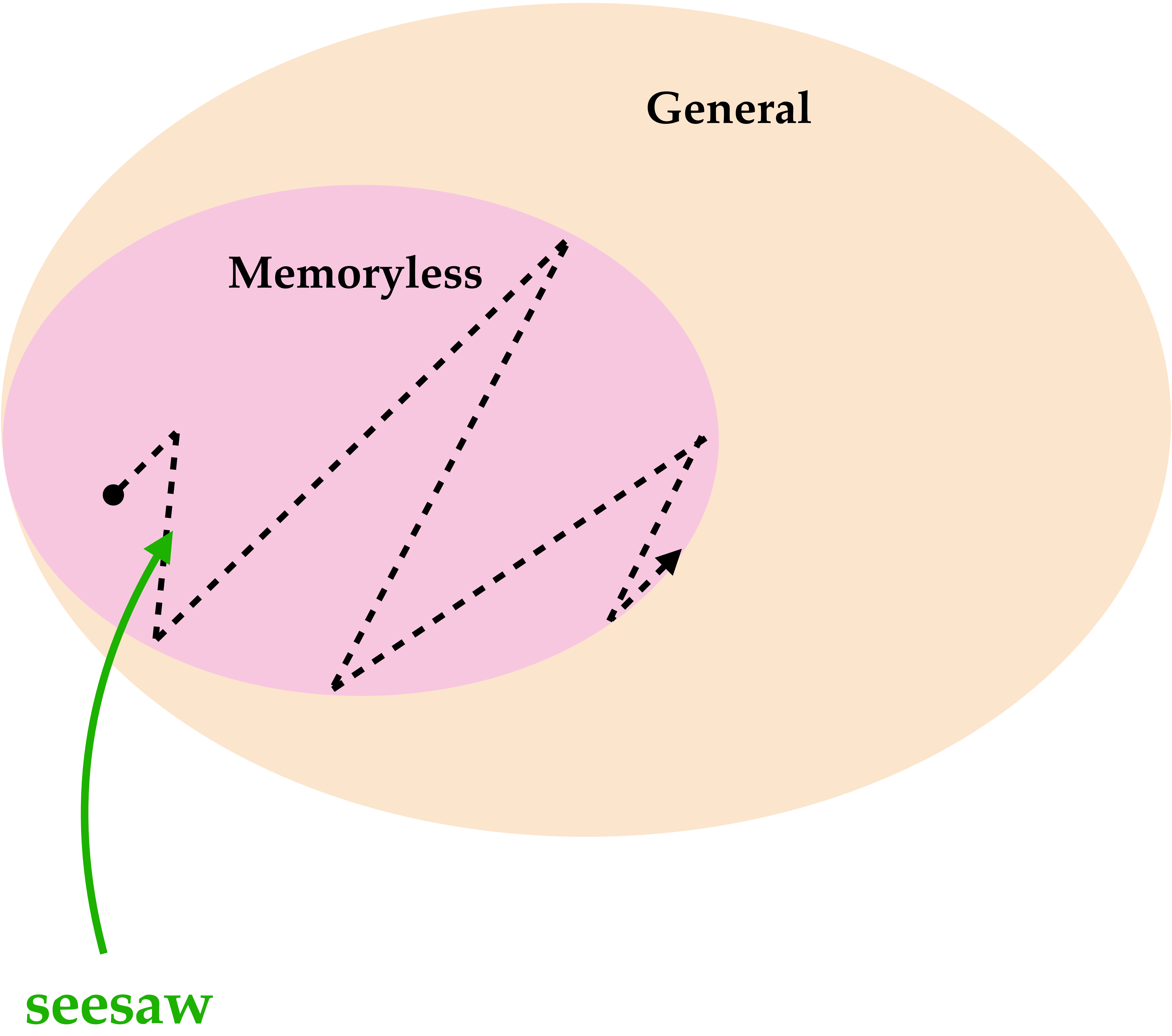
end while

$P_{\text{seesaw}} \leftarrow P_2$

return $P, \rho, \{M_i\}$

II. Approximation: Attainability

Channel discrimination without memory



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  $\Delta \leftarrow |P_1 - P_2|$

end while

$P_{\text{seesaw}} \leftarrow P_2$

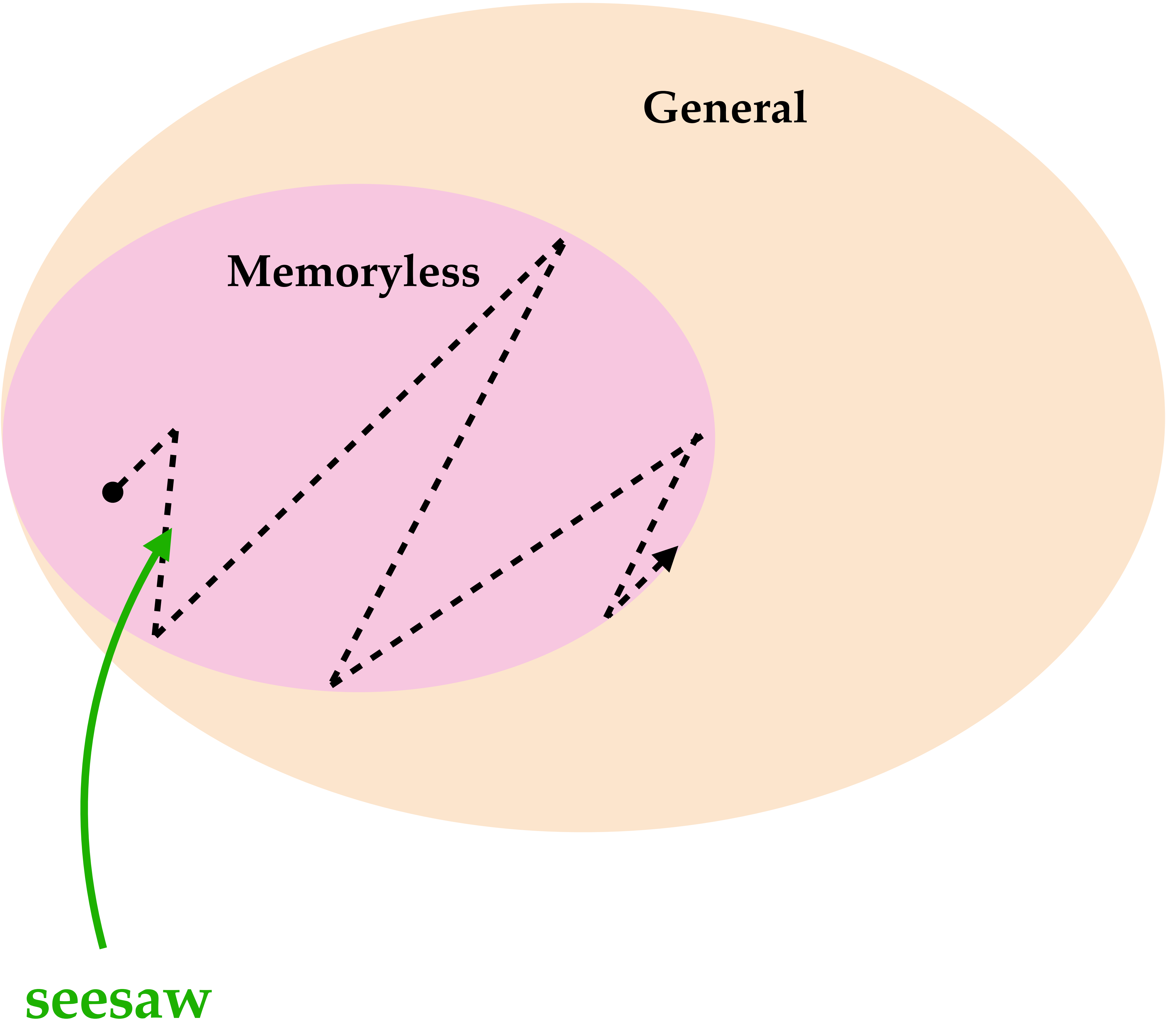
return $P, \rho, \{M_i\}$

$\boxed{P_{\text{seesaw}} \leq P_{\text{ML}}}$

lower bound

II. Approximation: Attainability

Channel discrimination without memory



Algorithm 1 Seesaw SDP algorithm

1: Initialize ρ
2: $\epsilon \leftarrow 10^{-5}$
3: $\Delta \leftarrow \infty$
4: **while** $\Delta > \varepsilon$ **do**
5: $\quad (P_1, \{M_i\}) \leftarrow \text{SDP } 1(\{p_i, C_i\}, \rho)$
6: $\quad (P_2, \rho) \leftarrow \text{SDP } 2(\{p_i, C_i\}, \{M_i\})$
7: $\quad \Delta \leftarrow |P_1 - P_2|$
8: **end while**
9: $P_{\text{seesaw}} \leftarrow P_2$
10: **return** $P, \rho, \{M_i\}$

$\boxed{P_{\text{seesaw}} \leq P_{\text{ML}}}$

lower bound

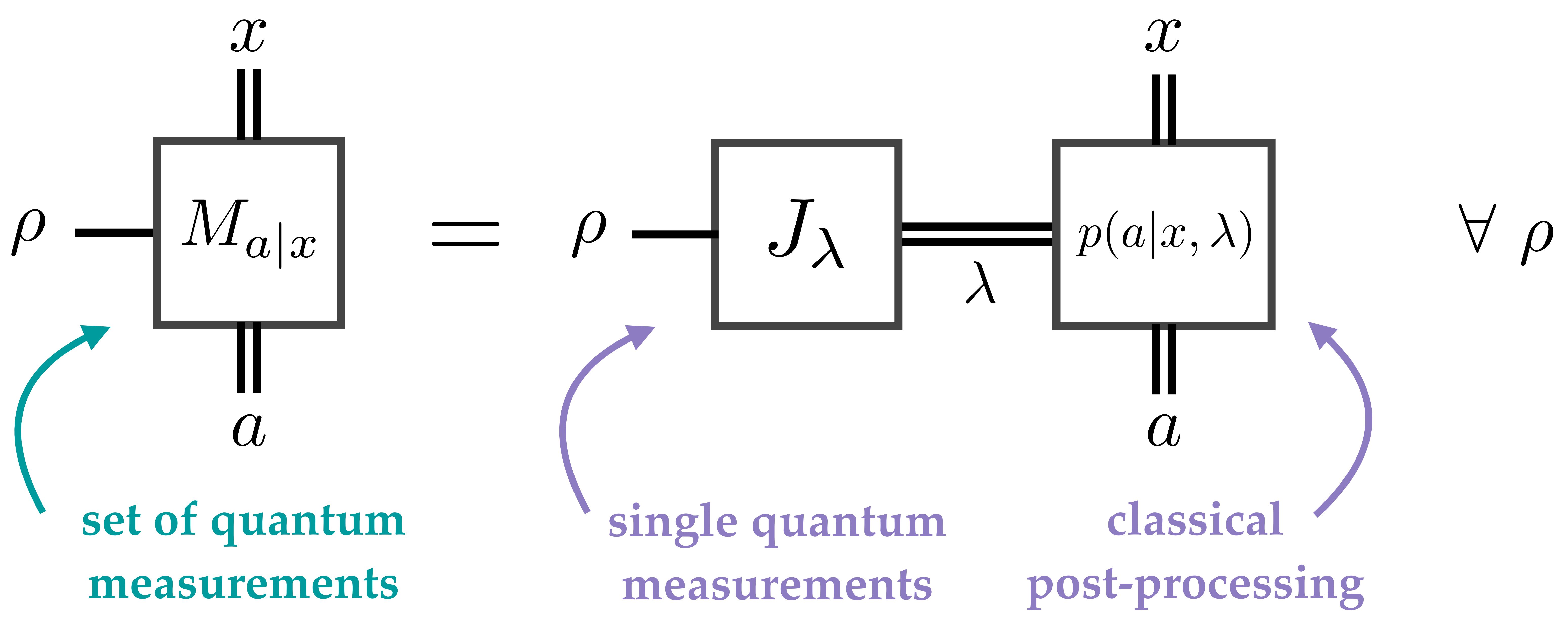
II. Approximation: Attainability

Most incompatible set of quantum measurements

II. Approximation: Attainability

Most incompatible set of quantum measurements

compatible set of quantum measurements:



II. Approximation: Attainability

Most incompatible set of quantum measurements

Question. What is the most incompatible set of N d-dimensional measurements with k outcomes?

II. Approximation: Attainability

Most incompatible set of quantum measurements

Question. What is the most incompatible set of N d-dimensional measurements with k outcomes?

$$\boxed{\tau(N,k,d) = \max_{\{M_{a|x}\}} t^*(\{M_{a|x}\})}$$

$N =$ number of measurements

$k =$ number of outcomes

$d =$ dimension

given $\{M_{a|x}\}$

$$t^*(\{M_{a|x}\}) = \min \quad t$$

$$\text{s.t.} \quad \sum_\mu D(a|x,\mu)\overline{G}_\mu - M_{a|x} \geq 0$$

$$t \geq 0$$

$$\overline{G}_\mu \geq 0, \quad \sum_\mu \overline{G}_\mu = (t+1)\mathbb{1}$$

II. Approximation: Attainability

Most incompatible set of quantum measurements

$N =$ number of measurements
$k =$ number of outcomes
$d =$ dimension

$$\text{given} \quad N, k, d$$

$$\tau(N,k,d) = \max_{\{M_{a|x}\}} \min_{t,\{\overline{G}_\mu\}} t$$

$$\text{s.t.} \quad \sum_\mu D(a|x,\mu)\overline{G}_\mu - M_{a|x} \geq 0$$

$$t \geq 0$$

$$\overline{G}_\mu \geq 0, \quad \sum_\mu \overline{G}_\mu = (t+1)\mathbb{1}$$

$$M_{a|x} \geq 0, \quad \sum_a M_{a|x} = \mathbb{1}$$

can I tackle this with a seesaw algorithm?

II. Approximation: Attainability

Most incompatible set of quantum measurements

Seesaw algorithm

given $\{M_{a|x}\}$

$$t_1^* = \min_{t,\{\overline{G}_\mu\}} \ t$$

$$\text{s.t.} \quad \sum_\mu D(a|x,\mu)\overline{G}_\mu - M_{a|x} \geq 0$$

$$t \geq 0$$

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II. Approximation: Attainability

Most incompatible set of quantum measurements

Seesaw algorithm

given $\{M_{a|x}\}$

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$$t \geq 0$$

$$\overline{G}_\mu \geq 0, \quad \sum_\mu \overline{G}_\mu = (t+1)\mathbb{1}$$

given $t, \{\overline{G}_\mu\}$

$$t_2^* = \max_{\{M_{a|x}\}} \ t$$

$$\text{s.t.} \quad \sum_\mu D(a|x,\mu)\overline{G}_\mu - M_{a|x} \geq 0$$

$$M_{a|x} \geq 0, \quad \sum_a M_{a|x} = \mathbb{1}$$

II. Approximation: Attainability

Most incompatible set of quantum measurements

Seesaw algorithm

given $\{M_{a|x}\}$

$$t_1^* = \min_{t,\{\overline{G}_\mu\}} t$$

$$\text{s.t.} \quad \sum_\mu D(a|x,\mu)\overline{G}_\mu - M_{a|x} \geq 0$$

$$t \geq 0$$

$$\overline{G}_\mu \geq 0, \quad \sum_\mu \overline{G}_\mu = (t+1)\mathbb{1}$$

given $t, \{\overline{G}_\mu\}$ → **constant**

$$t_2^* = \max_{\{M_{a|x}\}} t \longrightarrow \textbf{constant}$$

$$\text{s.t.} \quad \sum_\mu D(a|x,\mu)\overline{G}_\mu - M_{a|x} \geq 0$$

$$M_{a|x} \geq 0, \quad \sum_a M_{a|x} = \mathbb{1}$$

seesaw doesn't go anywhere

II. Approximation: Attainability

Most incompatible set of quantum measurements

Seesaw algorithm

N = number of measurements
k = number of outcomes
d = dimension

given $\quad N, k, d$

$$\tau(N,k,d) = \max_{\{M_{a|x}\}} \; \min_{t,\{\overline{G}_\mu\}} \; t$$

$$\text{s.t.} \quad \sum_\mu D(a|x,\mu)\overline{G}_\mu - M_{a|x} \geq 0$$

$$t \geq 0$$

$$\overline{G}_\mu \geq 0, \quad \sum_\mu \overline{G}_\mu = (t+1)\mathbb{1}$$

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II. Approximation: Attainability

Most incompatible set of quantum measurements

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$$M_{a|x} \geq 0, \quad \sum_a M_{a|x} = \mathbb{1}$$

$$\boxed{-\max_x \min_y f(x,y) = \min_x \max_y f(x,y)}$$

duality theory

II. Approximation: Attainability

Most incompatible set of quantum measurements

given $\{M_{a|x}\}$

$$t^* = \min_{t,\{\overline{G}_\mu\}} \; t$$

$$\text{s.t.} \quad \sum_\mu D(a|x,\mu)\overline{G}_\mu - M_{a|x} \geq 0$$

$$t \geq 0$$

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II. Approximation: Attainability

Most incompatible set of quantum measurements

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$$t \geq 0$$

$$\overline{G}_\mu \geq 0, \quad \sum_\mu \overline{G}_\mu = (t+1)\mathbb{1}$$

Dual problem $\longrightarrow$

given $\{M_{a|x}\}$

$$t^* = \max_{\Gamma,\{F_{a|x}\}} \sum_{ax} \mathrm{tr}\big(F_{a|x}M_{a|x}\big) - \mathrm{tr}(\Gamma)$$

$$\text{s.t.} \quad \sum_{ax} D(a|x,\mu)F_{a|x} \leq \Gamma$$

$$\mathrm{tr}(\Gamma) \leq 1$$

II. Approximation: Attainability

Most incompatible set of quantum measurements

Original

$$\text{given} \quad N, k, d$$

$$\tau(N,k,d) = \max_{\{M_{a|x}\}} \min_{t,\{\overline{G}_\mu\}} t$$

$$\text{s.t.} \quad \sum_\mu D(a|x,\mu)\overline{G}_\mu - M_{a|x} \geq 0$$

$$t \geq 0$$

$$\overline{G}_\mu \geq 0, \quad \sum_\mu \overline{G}_\mu = (t+1)\mathbb{1}$$

$$M_{a|x} \geq 0, \quad \sum_a M_{a|x} = \mathbb{1}$$

$$\Longleftrightarrow$$

Rewritten

$$\text{given} \quad N, k, d$$

$$\tau(N,k,d) = \max_{\{M_{a|x}\}} \max_{\Gamma,\{F_{a|x}\}} \sum_{ax} \mathrm{tr}\big(F_{a|x}M_{a|x}\big) + \mathrm{tr}(\Gamma)$$

$$\text{s.t.} \quad \sum_{ax} D(a|x,\mu)F_{a|x} \leq -\Gamma$$

$$\mathrm{tr}(\Gamma) \geq -1$$

$$M_{a|x} \geq 0, \quad \sum_a M_{a|x} = \mathbb{1}$$

II. Approximation: Attainability

Most incompatible set of quantum measurements

Original

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$$\tau(N,k,d) = \max_{\{M_{a|x}\}} \min_{t,\{\overline{G}_\mu\}} t$$

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$$M_{a|x} \geq 0, \quad \sum_a M_{a|x} = \mathbb{1}$$

$$\Longleftrightarrow$$

Rewritten

$$\text{given} \quad N, k, d$$

$$\tau(N,k,d) = \max_{\{M_{a|x}\}} \max_{\Gamma,\{F_{a|x}\}} \sum_{ax} \mathrm{tr}\left(F_{a|x}M_{a|x}\right) + \mathrm{tr}(\Gamma)$$

$$\text{s.t.} \quad \sum_{ax} D(a|x,\mu)F_{a|x} \leq -\Gamma$$

$$\mathrm{tr}(\Gamma) \geq -1$$

$$M_{a|x} \geq 0, \quad \sum_a M_{a|x} = \mathbb{1}$$

II. Approximation: Attainability

Most incompatible set of quantum measurements

Rewritten

$$\text{given} \quad N, k, d$$

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can I tackle this with a seesaw algorithm?

II. Approximation: Attainability

Most incompatible set of quantum measurements

Seesaw algorithm

$$\text{given} \quad \{M_{a|x}\}$$

$$\tau_1 = \max_{\Gamma, \{F_{a|x}\}} \sum_{ax} \mathrm{tr}\big(F_{a|x} M_{a|x}\big) + \mathrm{tr}(\Gamma)$$

$$\text{s.t.} \quad \sum_{ax} D(a|x,\mu) F_{a|x} \leq -\Gamma$$

$$\mathrm{tr}(\Gamma) \geq -1$$

$$\text{given} \quad \Gamma, \{F_{a|x}\}$$

$$\tau_2 = \max_{\{M_{a|x}\}} \sum_{ax} \mathrm{tr}\big(F_{a|x} M_{a|x}\big) + \mathrm{tr}(\Gamma)$$

$$\text{s.t.} \quad M_{a|x} \geq 0, \quad \sum_a M_{a|x} = \mathbb{1}$$

II. Approximation: Attainability

Most incompatible set of quantum measurements

Seesaw algorithm

$$\text{given} \quad \{M_{a|x}\}$$

$$\tau_1 = \max_{\Gamma, \{F_{a|x}\}} \sum_{ax} \mathrm{tr}\left(F_{a|x} M_{a|x}\right) + \mathrm{tr}(\Gamma)$$

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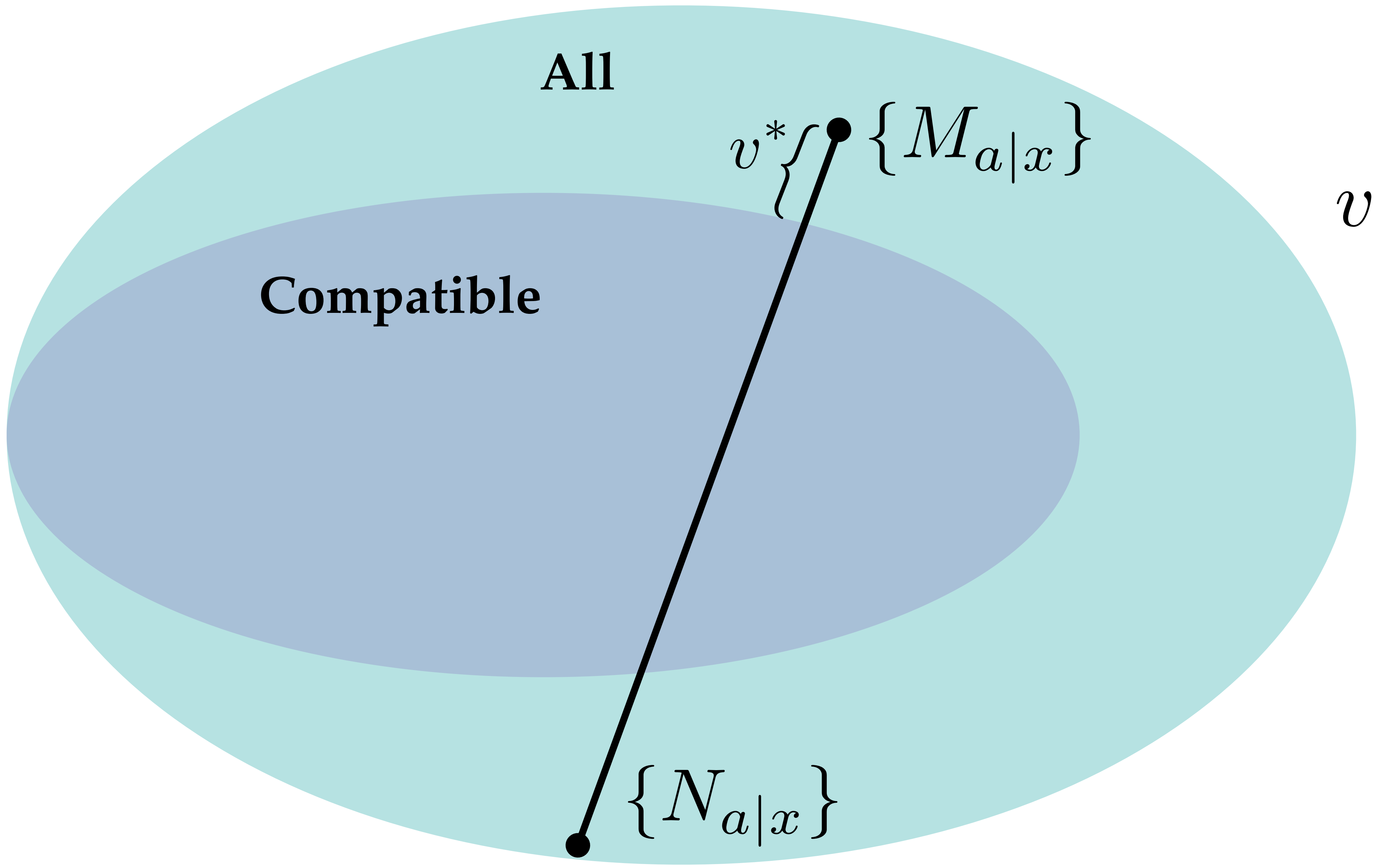
$$\text{s.t.} \quad M_{a|x} \geq 0, \quad \sum_a M_{a|x} = \mathbb{1}$$

II. Approximation: Attainability

Most incompatible set of quantum measurements

Geometric interpretation



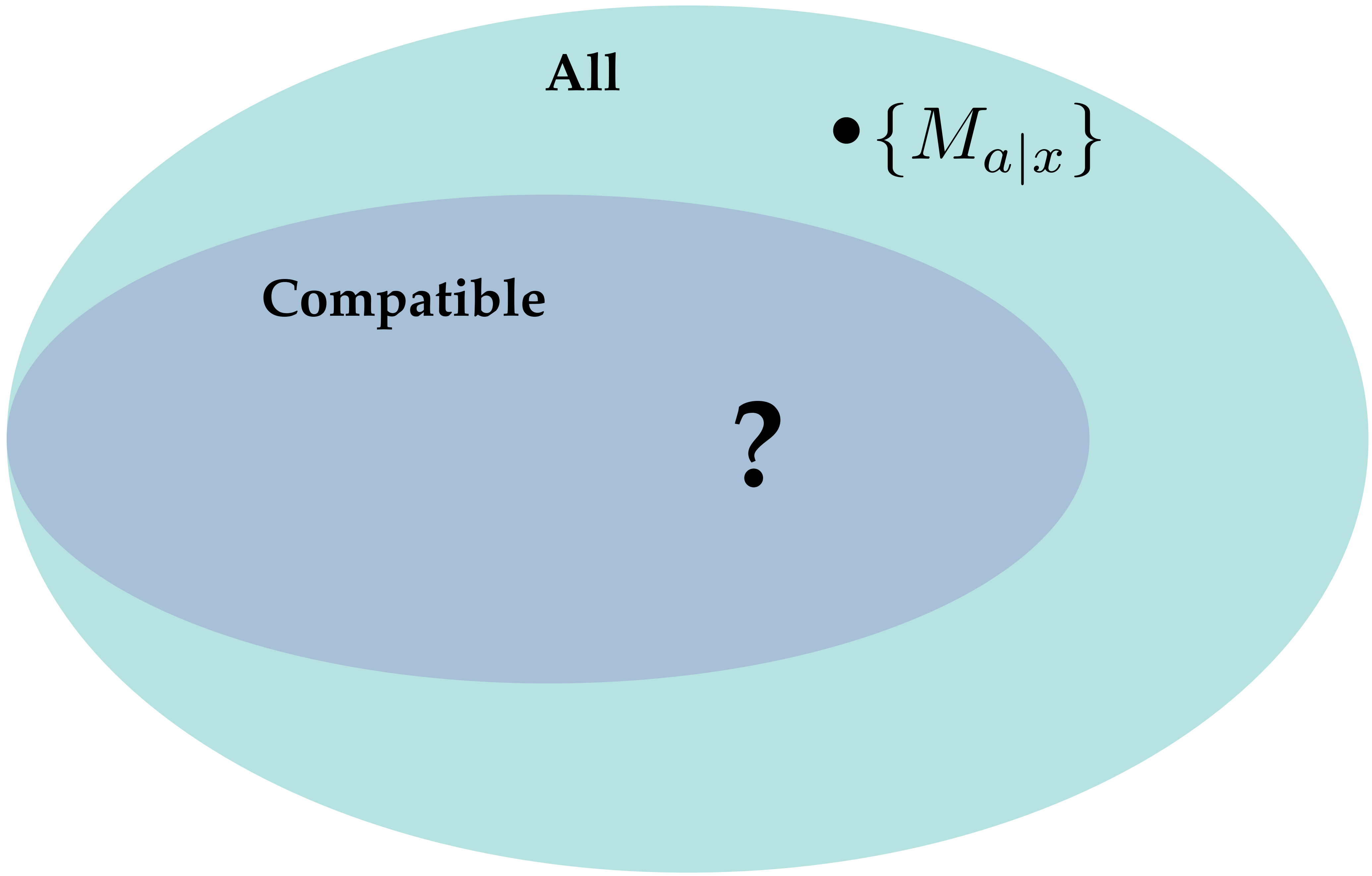
Primal problem

$$
\begin{aligned}
&\text{given } \{M_{a|x}\} \\
v^* = &\max \ v \\
&\text{s.t. } v\, M_{a|x} + (1-v) N_{a|x} = \sum_\mu D(a|x,\mu) G_\mu \\
&\quad 0 \le v \le 1 \\
&\quad N_{a|x} \ge 0, \quad \sum_a N_{a|x} = \mathbb{1} \\
&\quad G_\mu \ge 0, \quad \sum_\mu G_\mu = \mathbb{1}
\end{aligned}
$$

II. Approximation: Attainability

Most incompatible set of quantum measurements

Geometric interpretation



Dual problem

$$\text{given} \quad \{M_{a|x}\}$$

$$t^* = \max_{\Gamma, \{F_{a|x}\}} \sum_{ax} \mathrm{tr}\big(F_{a|x} M_{a|x}\big) - \mathrm{tr}(\Gamma)$$

$$\text{s.t.} \quad \sum_{ax} D(a|x,\mu) F_{a|x} \leq \Gamma$$

$$\mathrm{tr}(\Gamma) \leq 1$$

II. Approximation: Attainability

Most incompatible set of quantum measurements

$$t^* = \max\left\{t \;\middle|\; \frac{M_{a|x} + tN_{a|x}}{t+1} = \sum_\mu D(a|x,\mu)G_\mu\right\}$$

$$t^* = \sum_{ax} \mathrm{tr}\left(F^*_{a|x} M_{a|x}\right) - \mathrm{tr}(\Gamma^*) = 0 \iff \{M_{a|x}\} \text{ are compatible}$$

II. Approximation: Attainability

Most incompatible set of quantum measurements

$$t^* = \max\left\{ t \;\middle|\; \frac{M_{a|x} + tN_{a|x}}{t+1} = \sum_\mu D(a|x,\mu) G_\mu \right\}$$

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II. Approximation: Attainability

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$$t^* = \sum_{ax} \mathrm{tr}\left(F^*_{a|x} M_{a|x}\right) - \mathrm{tr}(\Gamma^*) > 0 \iff \{M_{a|x}\} \text{ are } \textbf{incompatible}$$

II. Approximation: Attainability

Most incompatible set of quantum measurements

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II. Approximation: Attainability

Most incompatible set of quantum measurements

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$$t^* = \sum_{ax} \mathrm{tr}\left(F^*_{a|x} M_{a|x}\right) - \mathrm{tr}(\Gamma^*) = 0 \iff \{M_{a|x}\} \text{ are compatible} \iff \sum_{ax} \mathrm{tr}\left(F^*_{a|x} M_{a|x}\right) = \mathrm{tr}(\Gamma^*)$$

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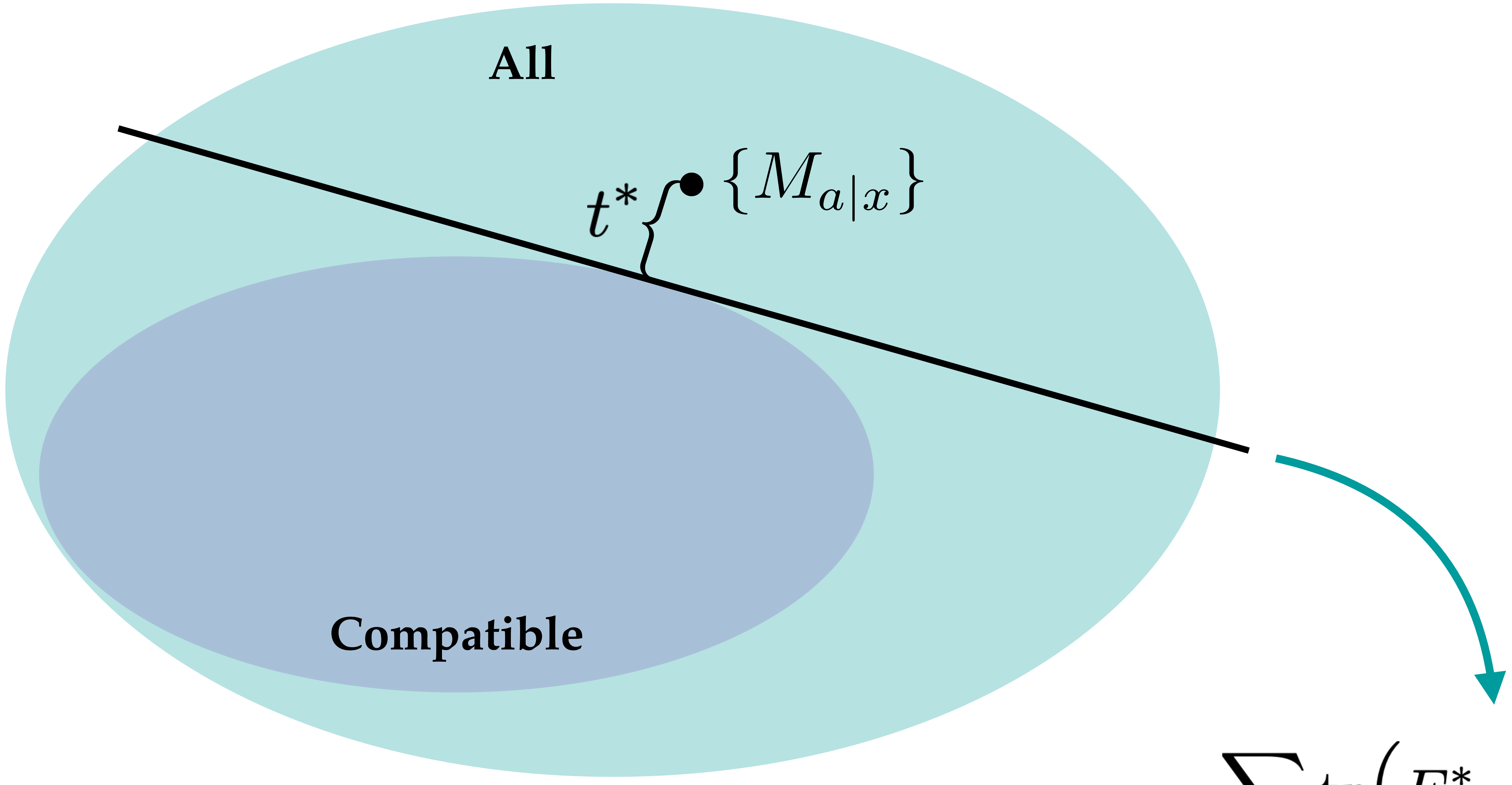
$$\boxed{\sum_{ax} \mathrm{tr}\left(F^*_{a|x} M_{a|x}\right) \leq \mathrm{tr}(\Gamma^*)}$$ **incompatibility witness**

II. Approximation: Attainability

Most incompatible set of quantum measurements



Dual problem

$$\text{given} \quad \{M_{a|x}\}$$

$$t^* = \max_{\Gamma,\{F_{a|x}\}} \sum_{ax} \mathrm{tr}\big(F_{a|x} M_{a|x}\big) - \mathrm{tr}(\Gamma)$$

$$\text{s.t.} \quad \sum_{ax} D(a|x,\mu) F_{a|x} \leq \Gamma$$

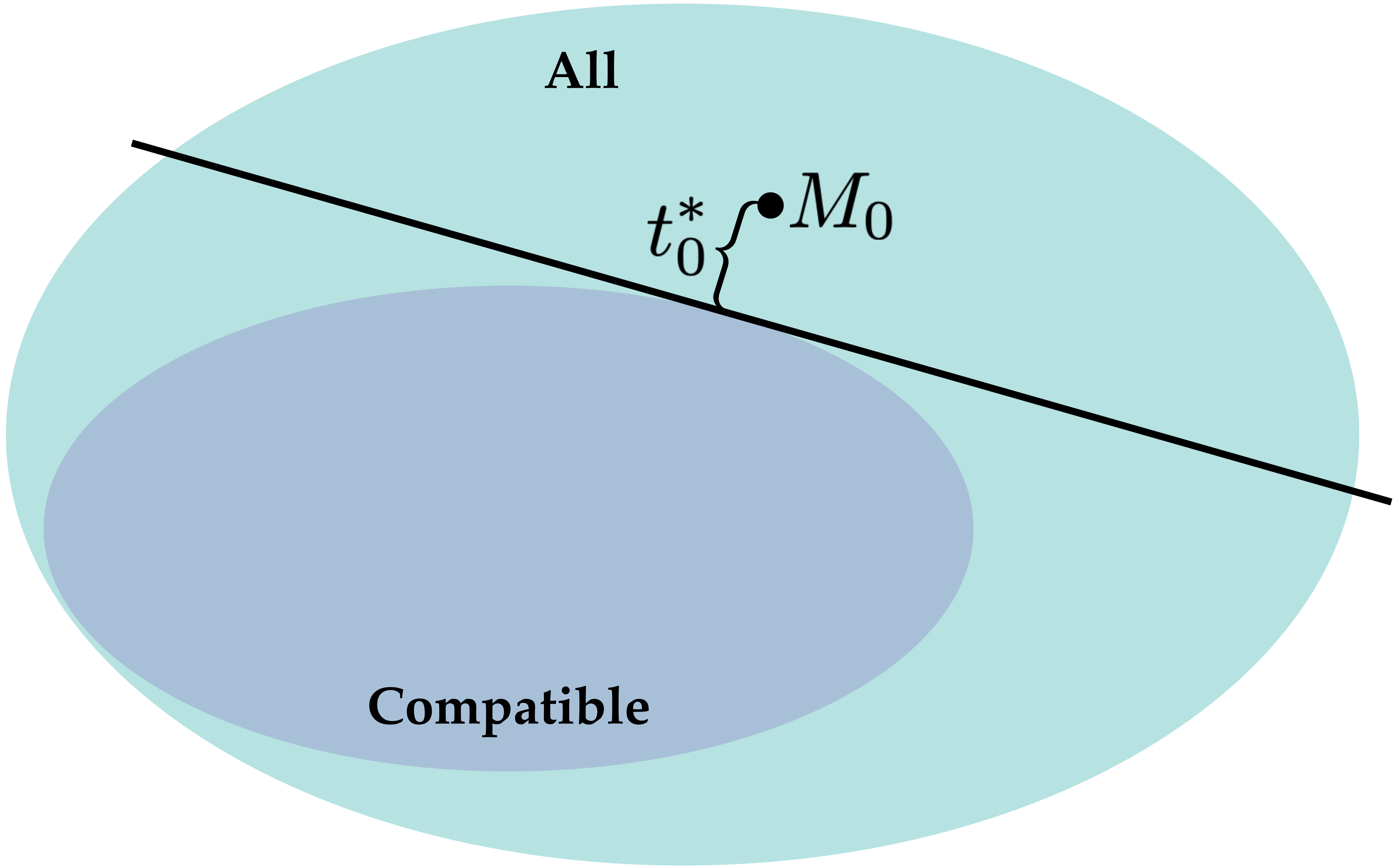
$$\mathrm{tr}(\Gamma) \leq 1$$

$$\sum_{ax} \mathrm{tr}\Big(F^*_{a|x} M_{a|x}\Big) = \mathrm{tr}(\Gamma^*)$$

II. Approximation: Attainability

Most incompatible set of quantum measurements

Seesaw algorithm



SDP 1

$$\text{given} \quad \{M_{a|x}\}$$

$$t^* = \max_{\Gamma, \{F_{a|x}\}} \sum_{ax} \mathrm{tr}\big(F_{a|x} M_{a|x}\big) + \mathrm{tr}(\Gamma)$$

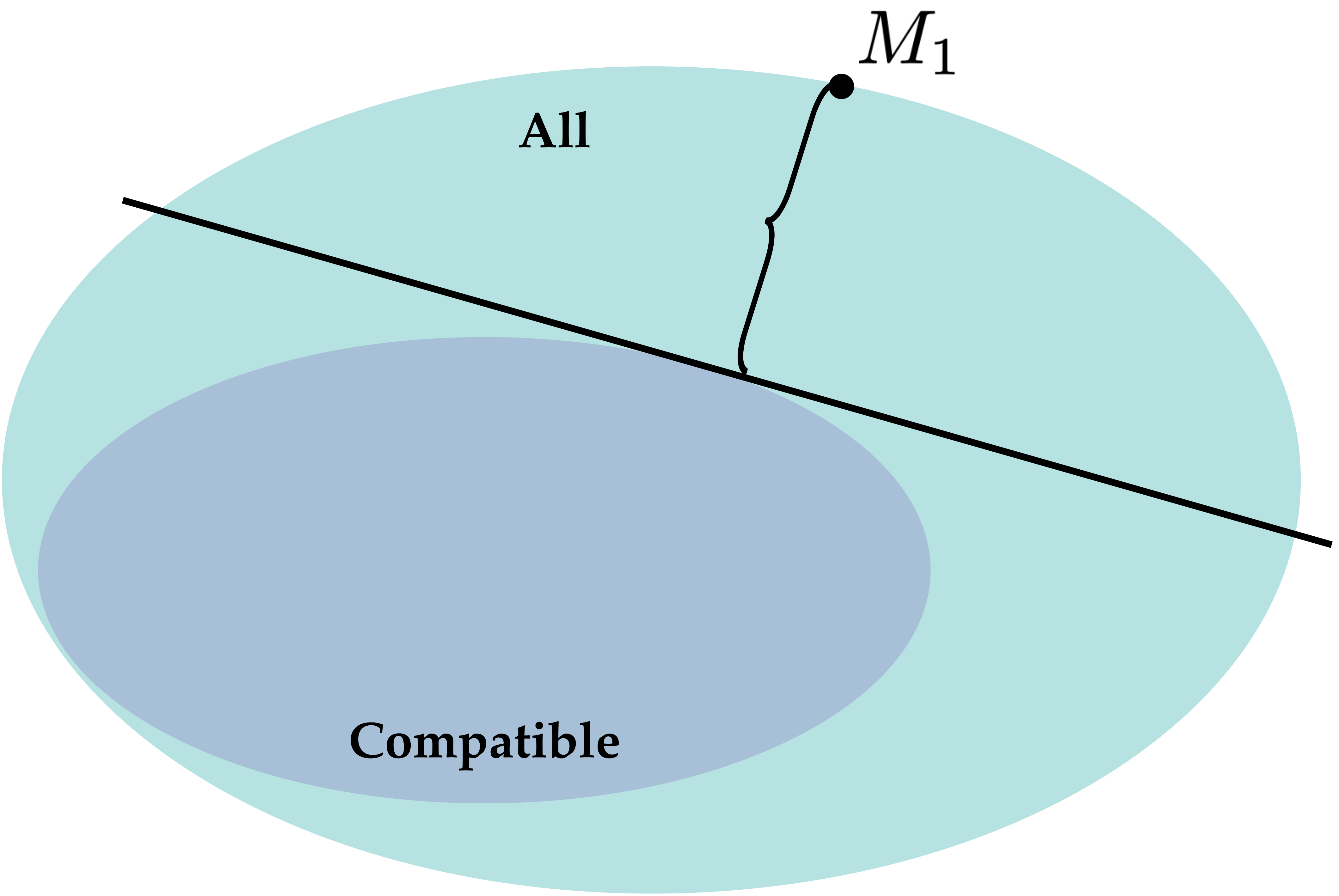
$$\text{s.t.} \quad \sum_{ax} D(a|x,\mu) F_{a|x} \leq -\Gamma$$

$$\mathrm{tr}(\Gamma) \geq -1$$

II. Approximation: Attainability

Most incompatible set of quantum measurements

Seesaw algorithm



SDP 2

$$\text{given} \quad \Gamma, \{F_{a|x}\}$$

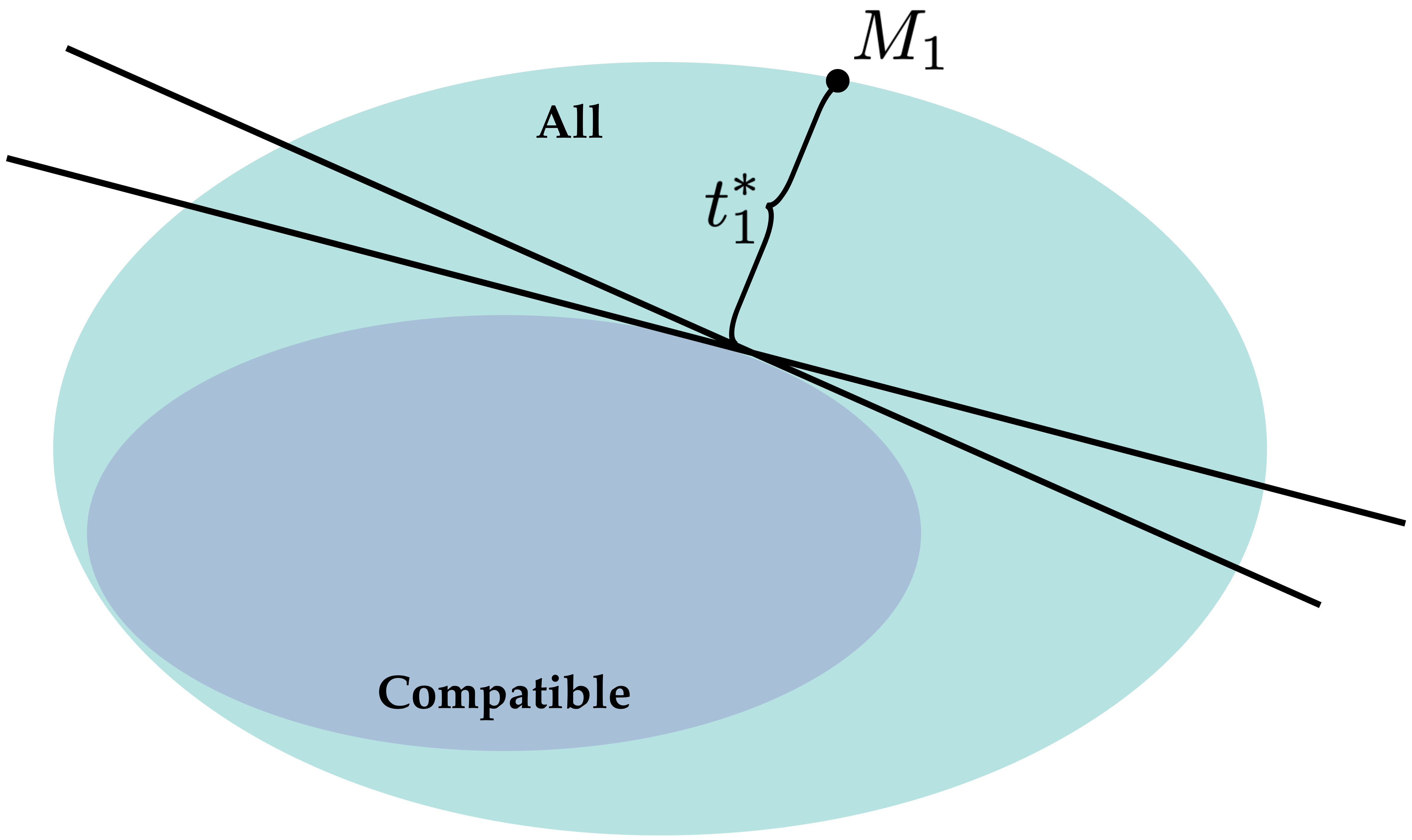
$$\max_{\{M_{a|x}\}} \sum_{ax} \mathrm{tr}\big(F_{a|x} M_{a|x}\big) + \mathrm{tr}(\Gamma)$$

$$\text{s.t.} \quad M_{a|x} \geq 0, \quad \sum_{a} M_{a|x} = \mathbb{1}$$

II. Approximation: Attainability

Most incompatible set of quantum measurements

Seesaw algorithm



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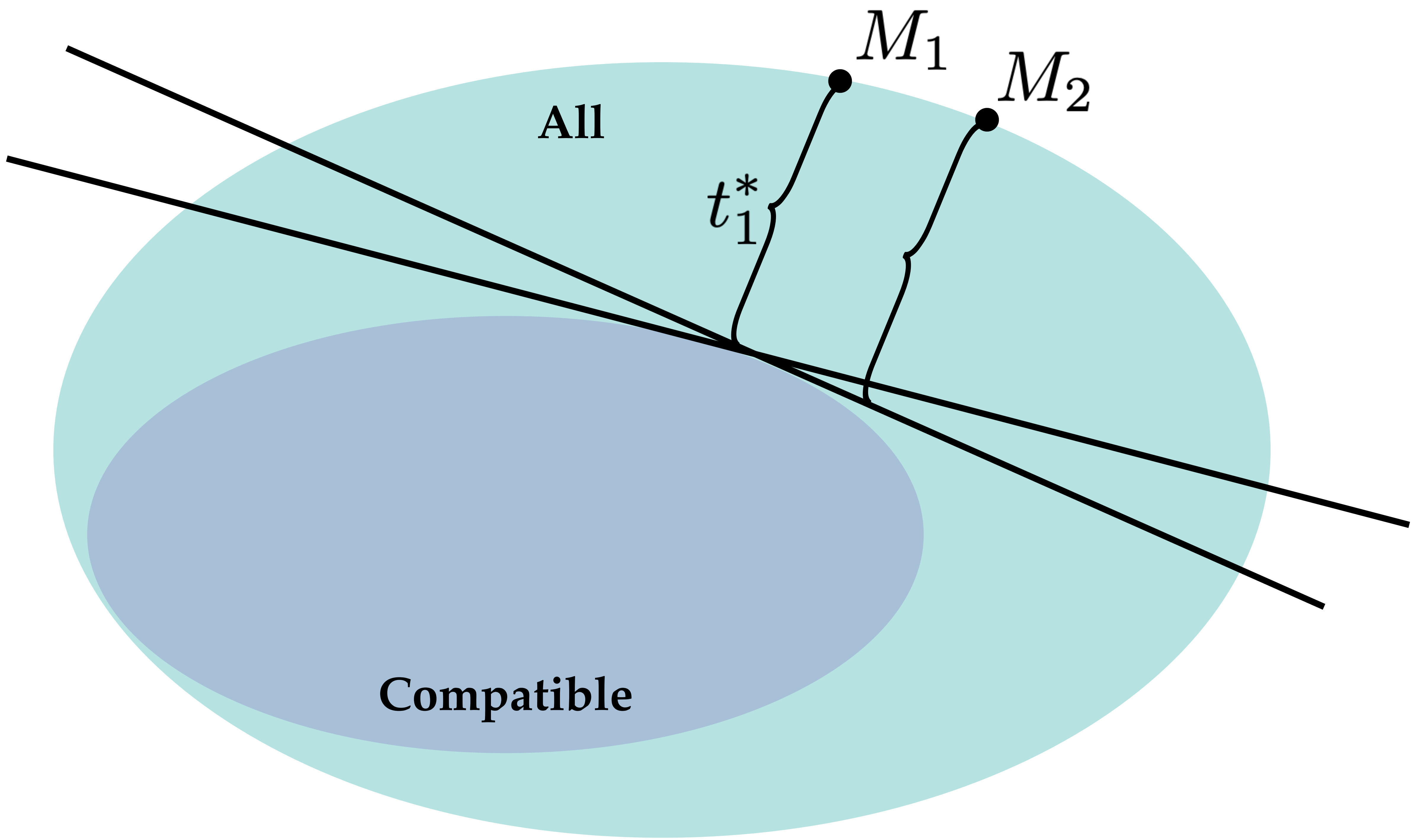
II. Approximation: Attainability

Most incompatible set of quantum measurements

Seesaw algorithm



SDP 2

$$\text{given} \quad \Gamma, \{F_{a|x}\}$$

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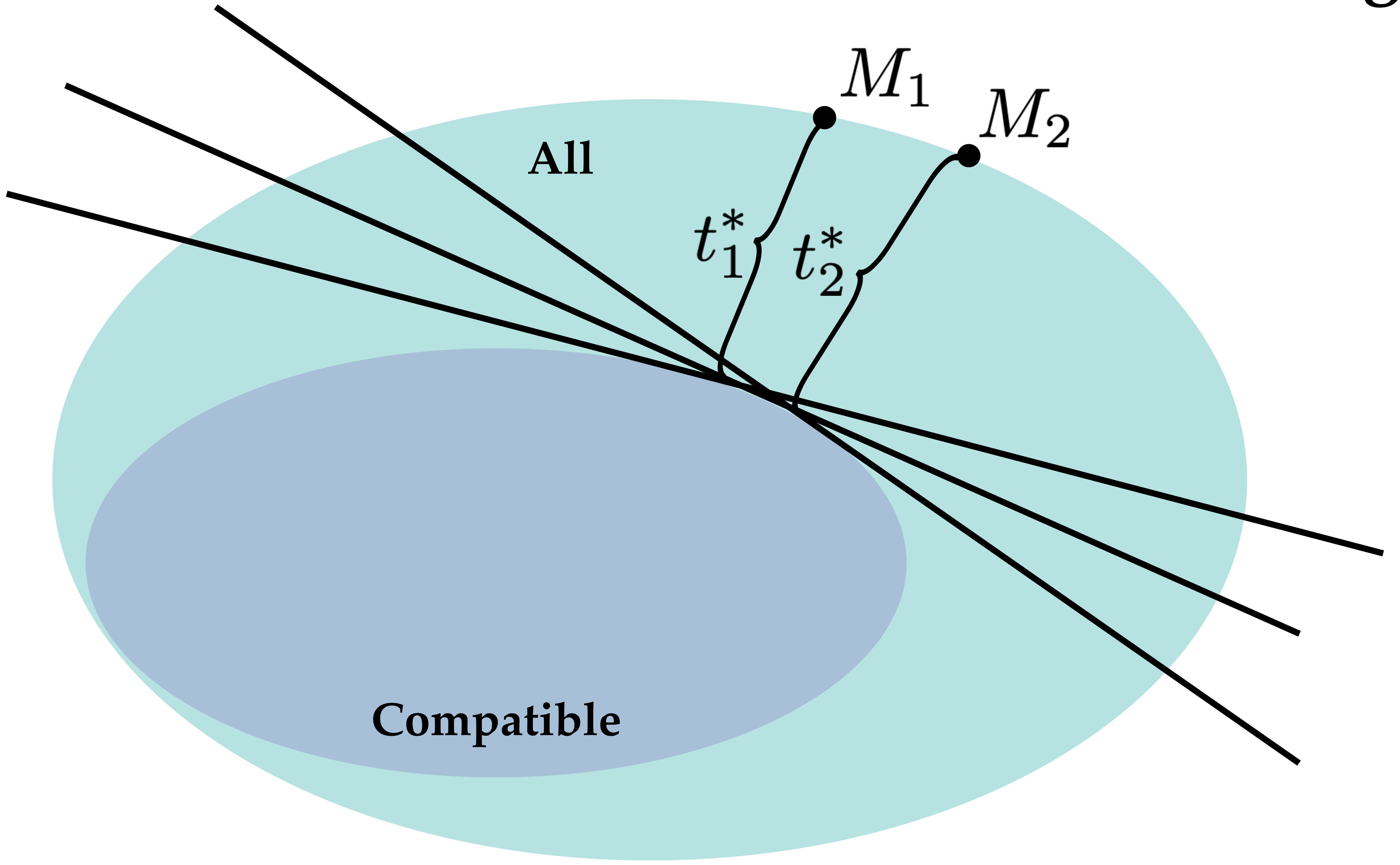
II. Approximation: Attainability

Most incompatible set of quantum measurements

Seesaw algorithm



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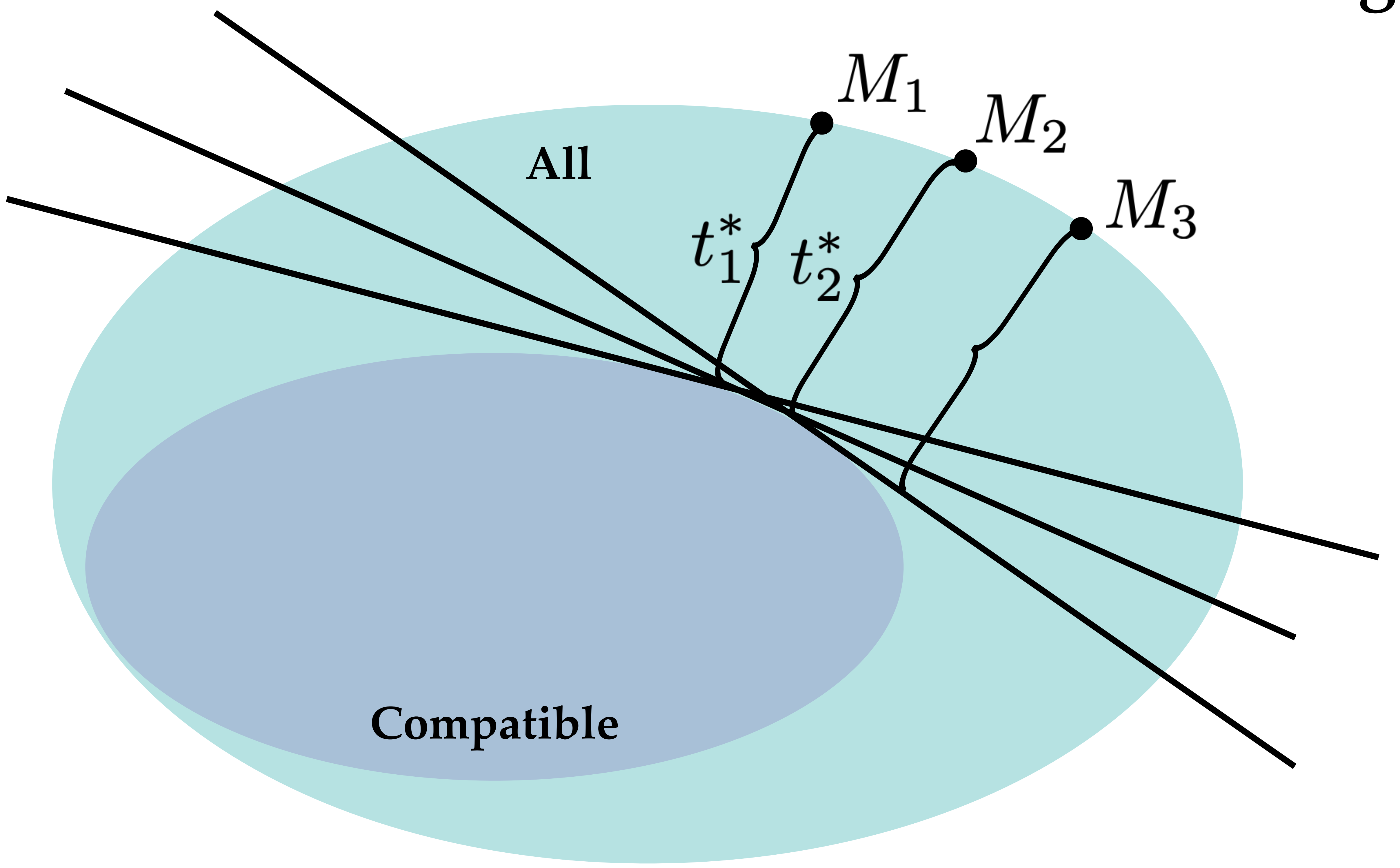
II. Approximation: Attainability

Most incompatible set of quantum measurements

Seesaw algorithm



SDP 2

$$\text{given} \quad \Gamma, \{F_{a|x}\}$$

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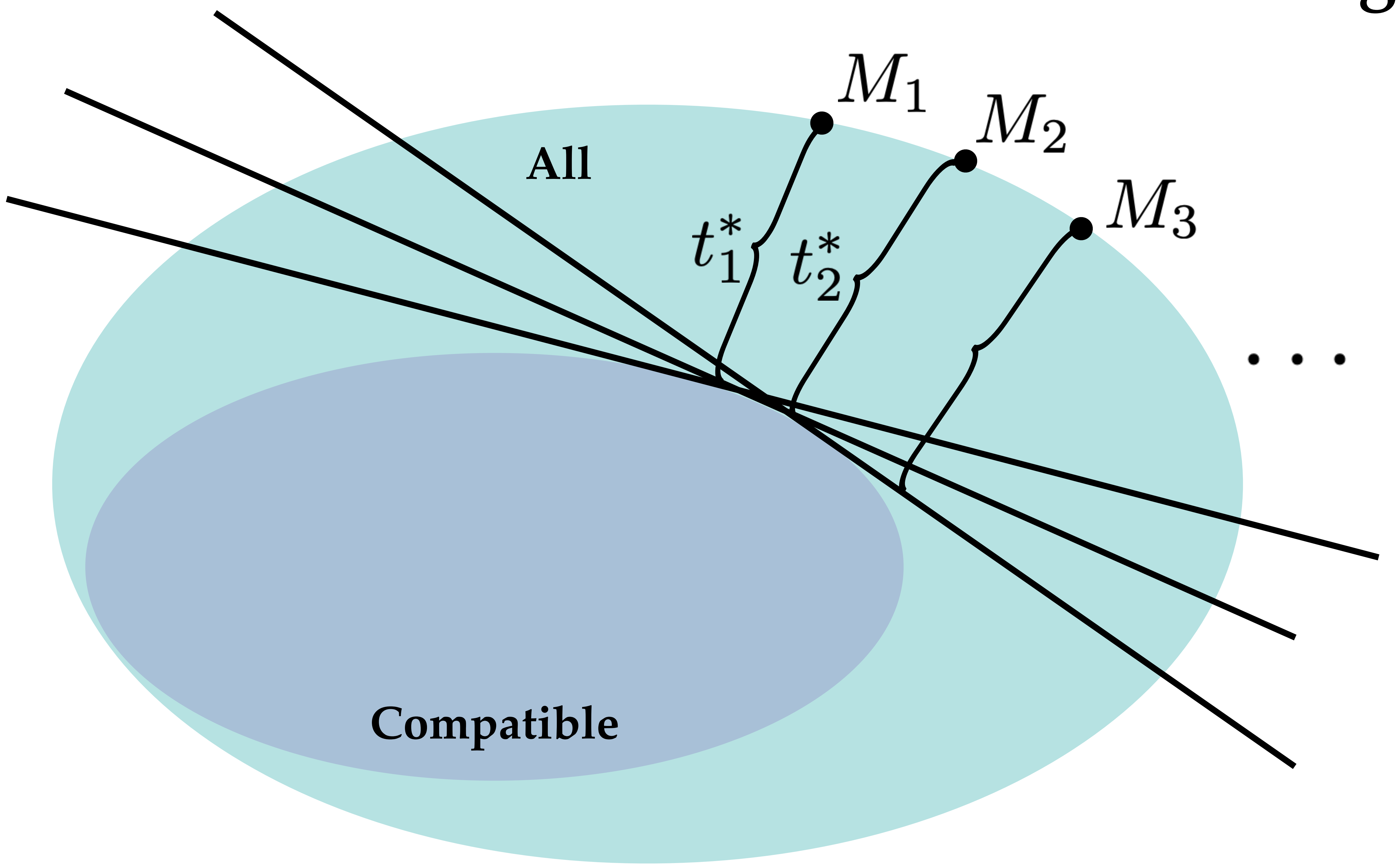
II. Approximation: Attainability

Most incompatible set of quantum measurements

Seesaw algorithm



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$$\text{s.t.} \quad M_{a|x} \geq 0, \quad \sum_a M_{a|x} = \mathbb{1}$$

seesaw goes somewhere!

II. Approximating: Attainability

MAIN MESSAGE:

Seesaws are useful tools to tackle multilinear problems

Duality theory can help you convert max-min problems into max-max problems and tackle it with seesaws.

II. Approximating: Attainability

OTHER SDP SEESAW APPLICATIONS

II. Approximating: Impossibility

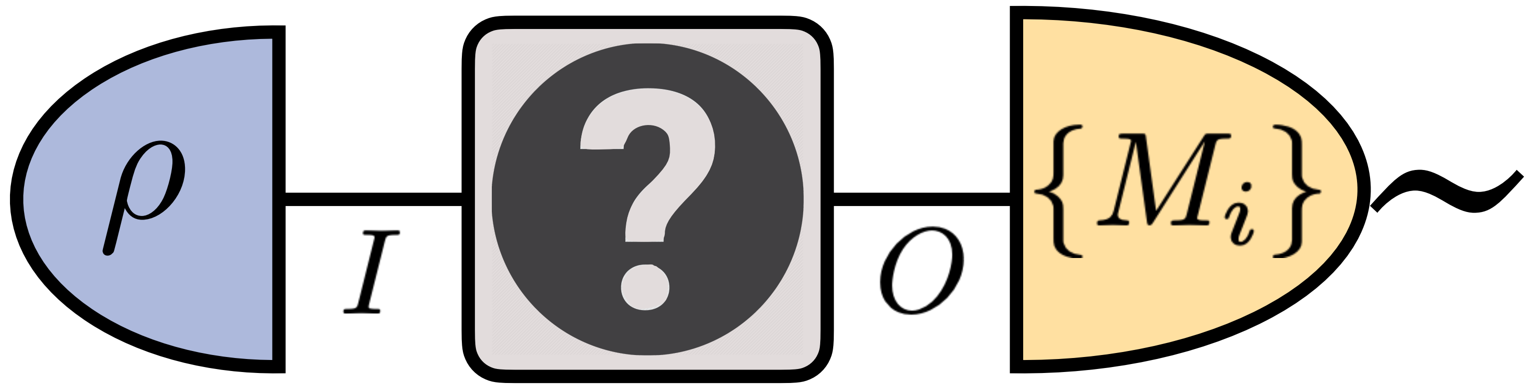
II. Approximation: Impossibility

Channel discrimination without memory

II. Approximation: Impossibility

Channel discrimination without memory

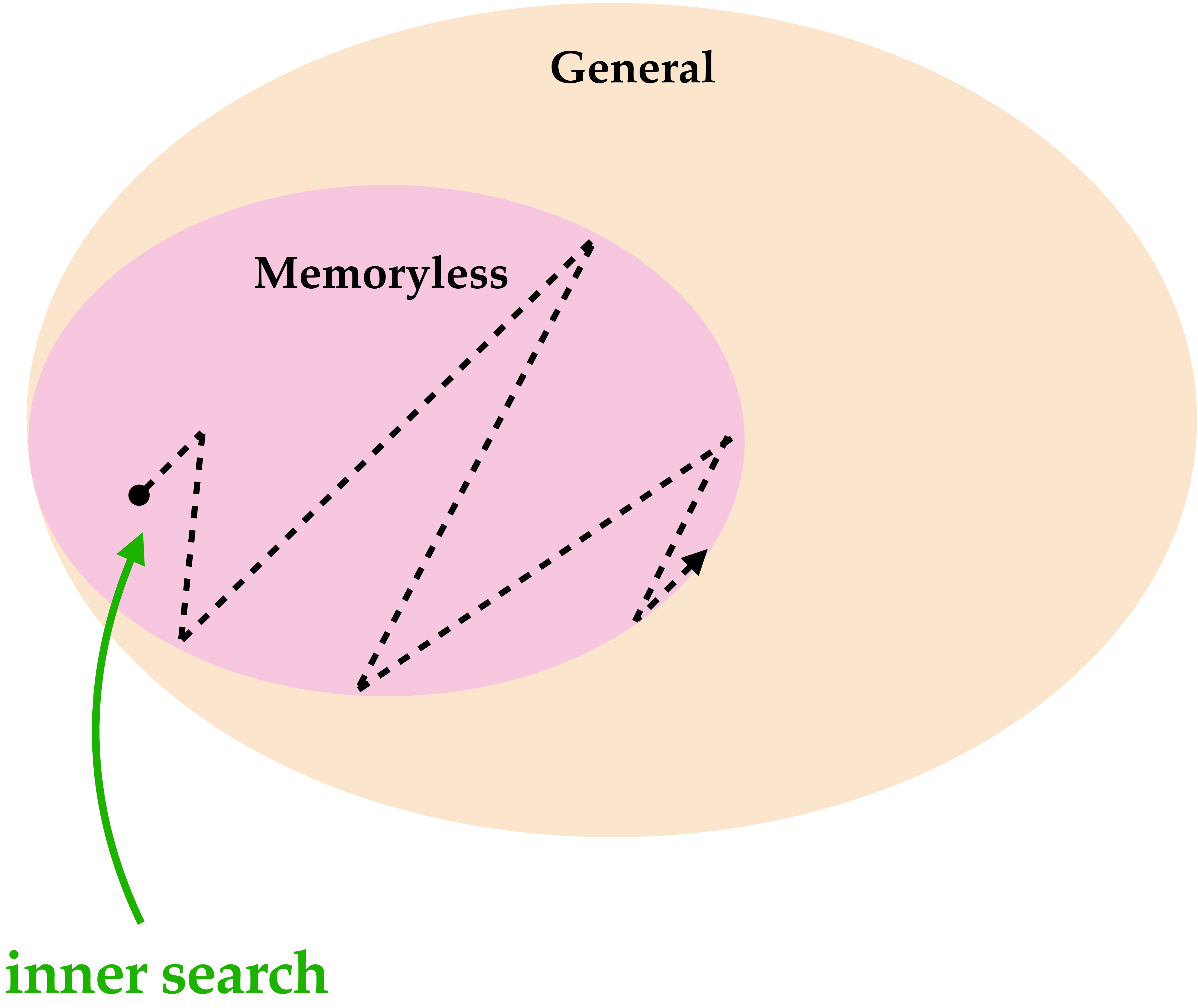
$$\mathcal{E} = \{p_i, \widetilde{C}_i\}$$

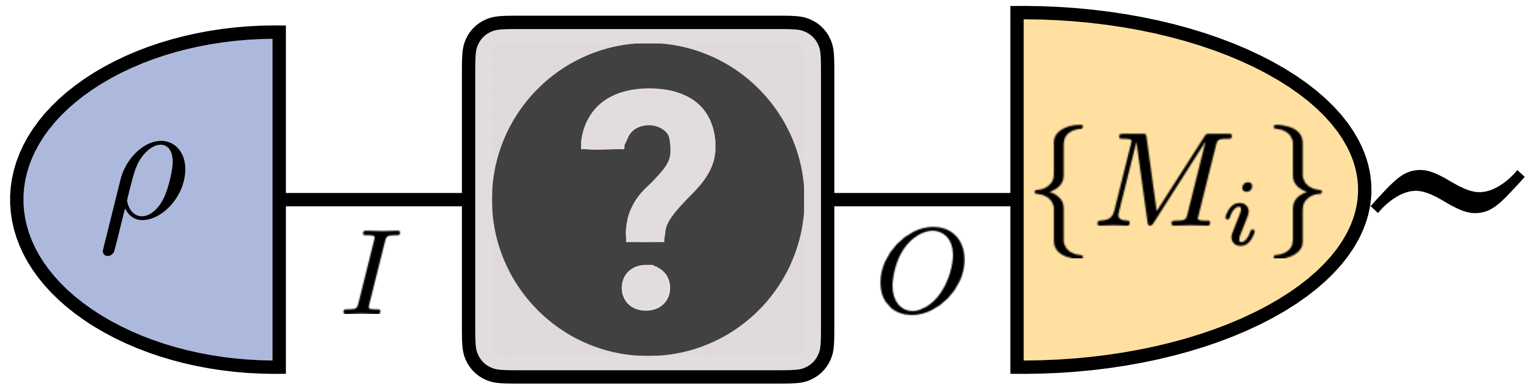


$$P_{\mathrm{ML}} = \max_{\rho, \{M_i\}} \sum_i p_i \operatorname{tr}\left[\widetilde{C}_i(\rho)\, M_i\right]$$

II. Approximation: Impossibility

Channel discrimination without memory

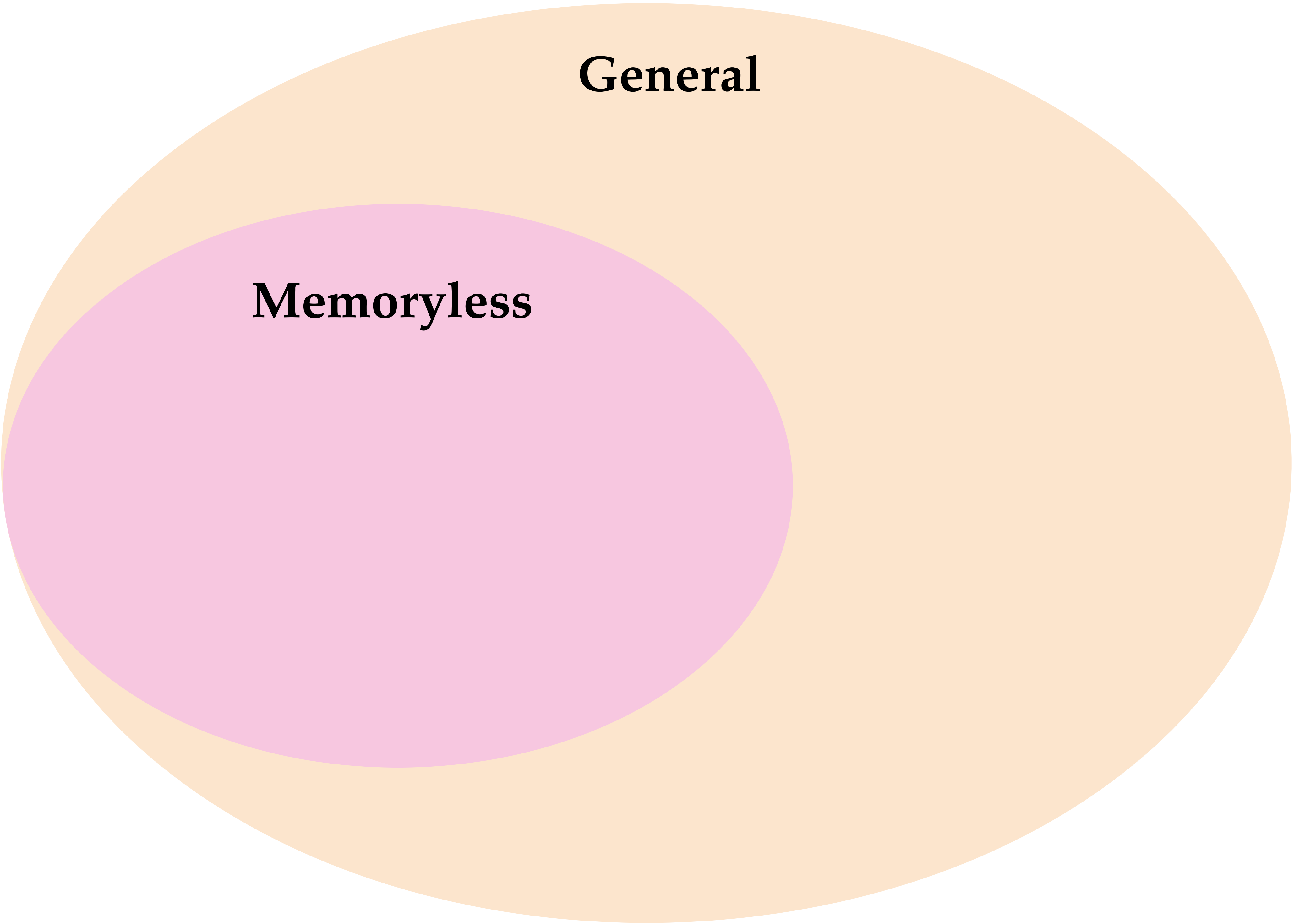


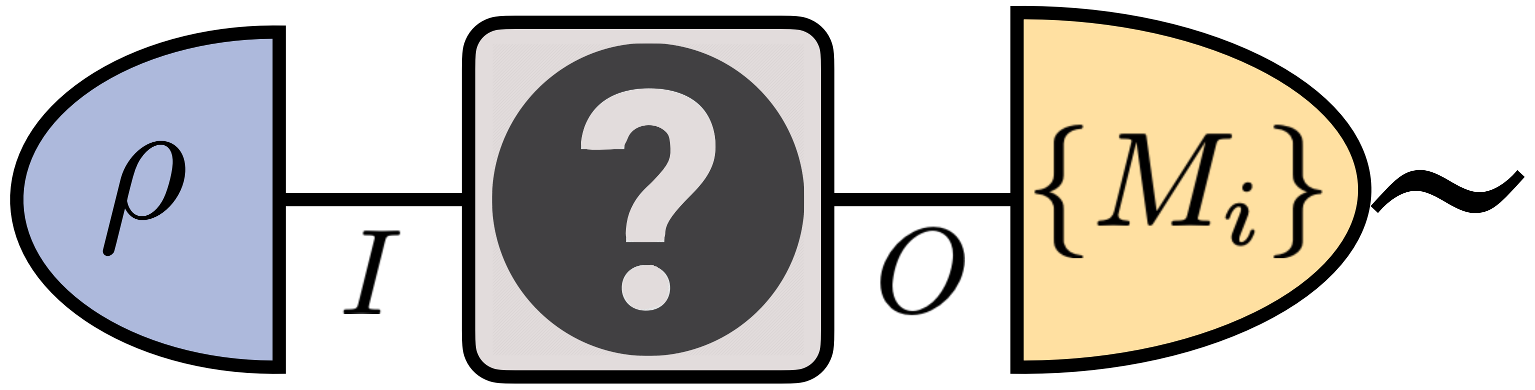


$$T_i = \rho \otimes M_i^T$$

II. Approximation: Impossibility

Channel discrimination without memory



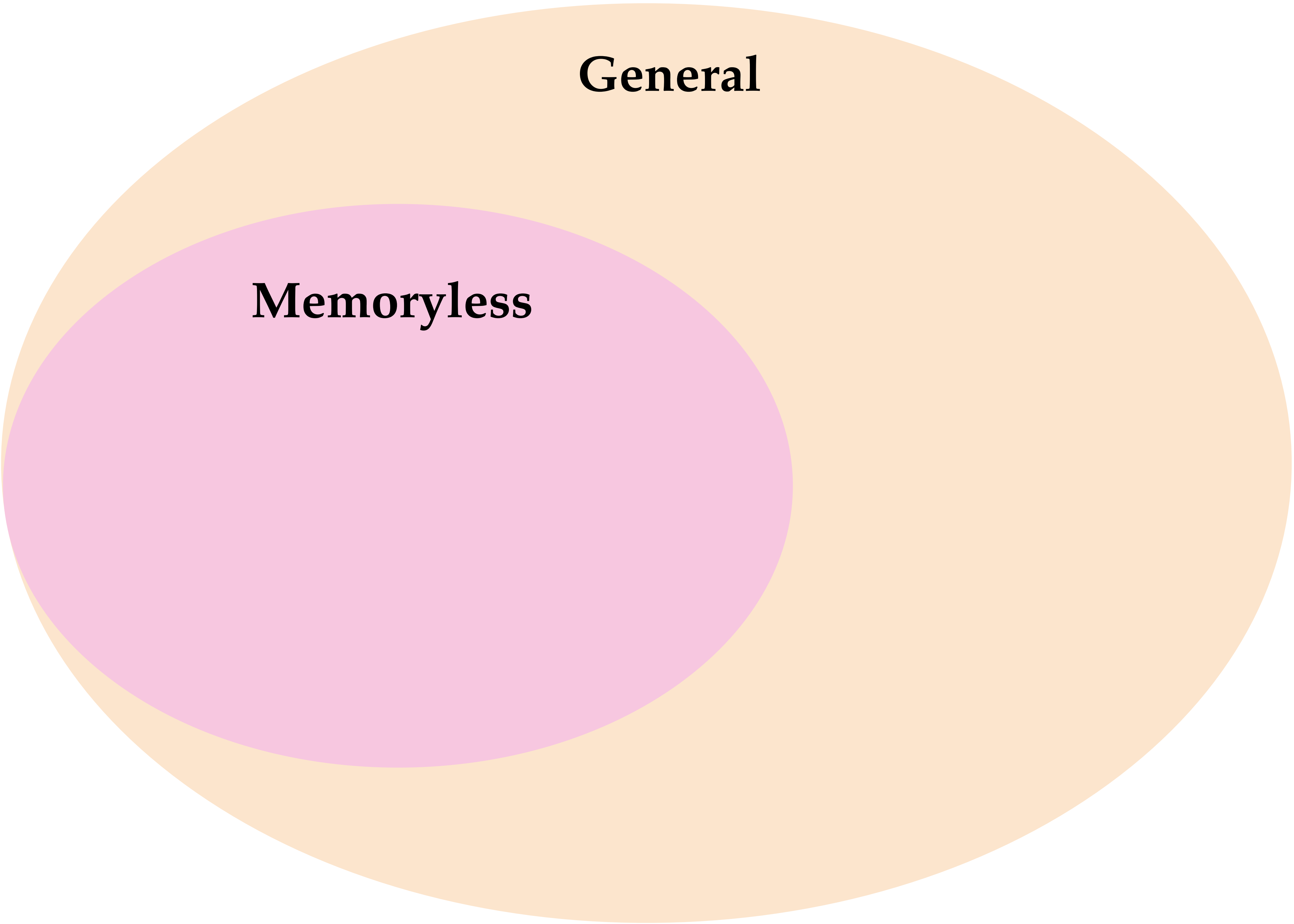


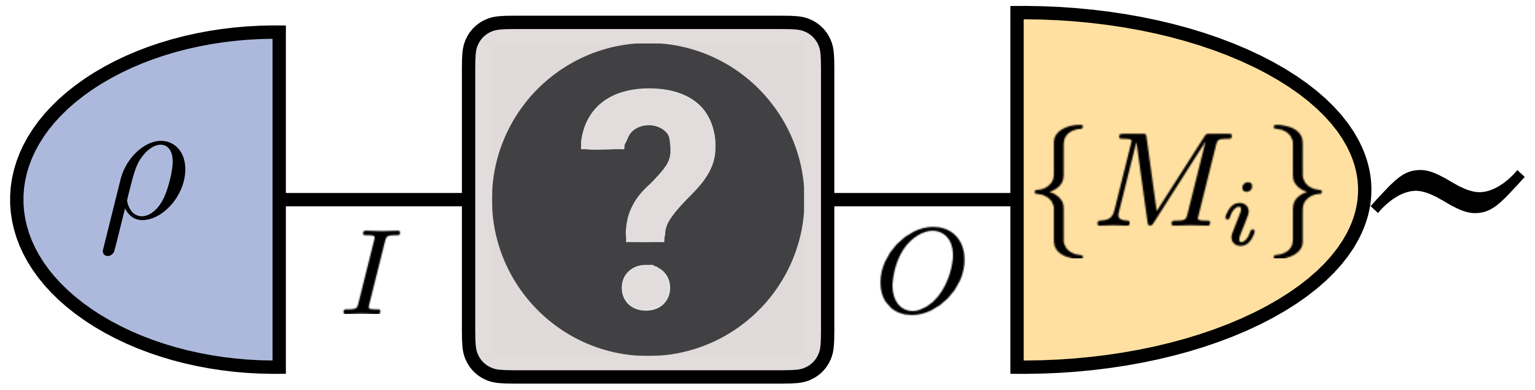
$$T_i = \rho \otimes M_i^T$$

How can I show that the optimal tester **requires** memory for implementation (does not accept such decomposition)?

II. Approximation: Impossibility

Channel discrimination without memory





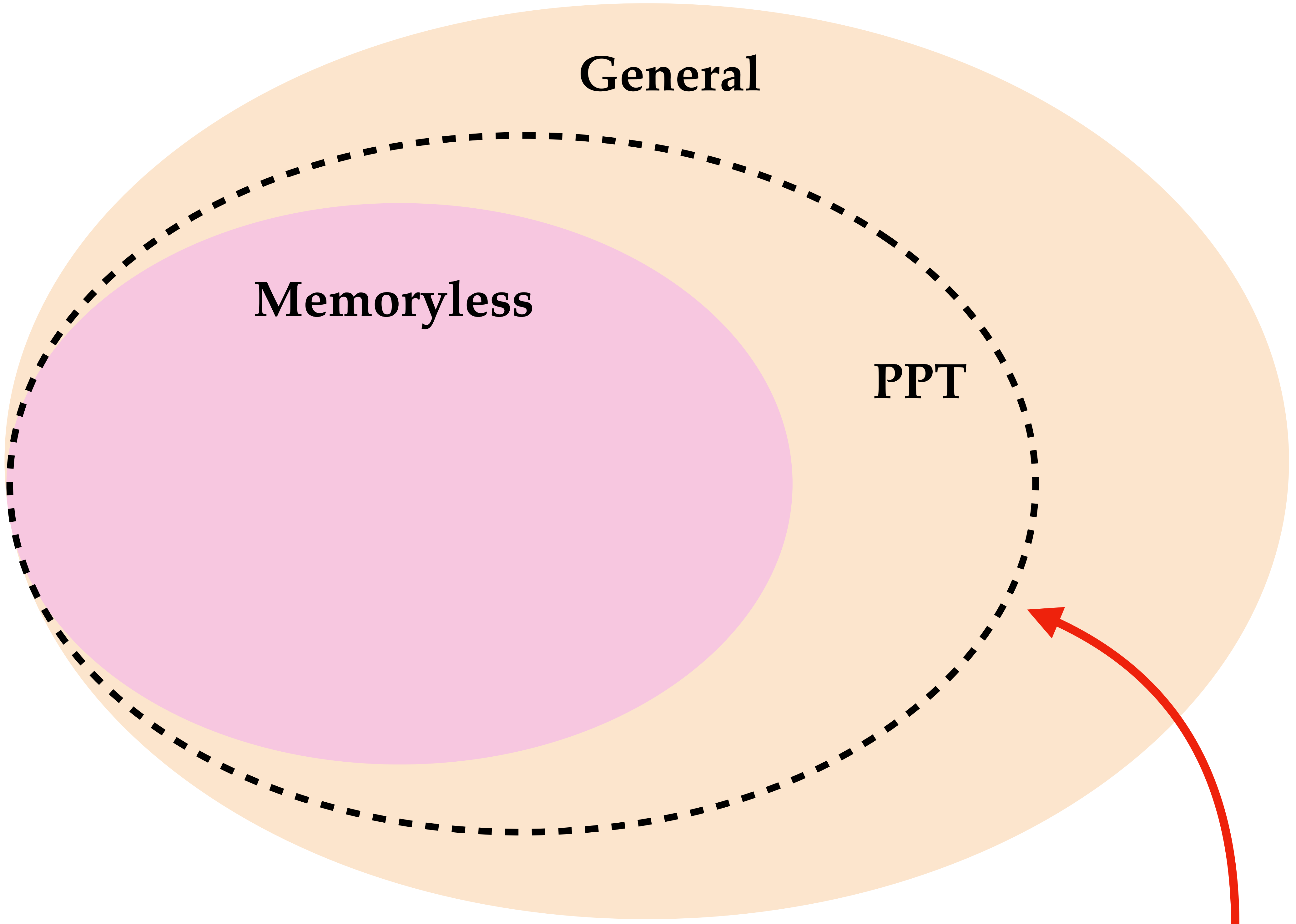
$$T_i = \rho \otimes M_i^T$$

$$\Downarrow$$

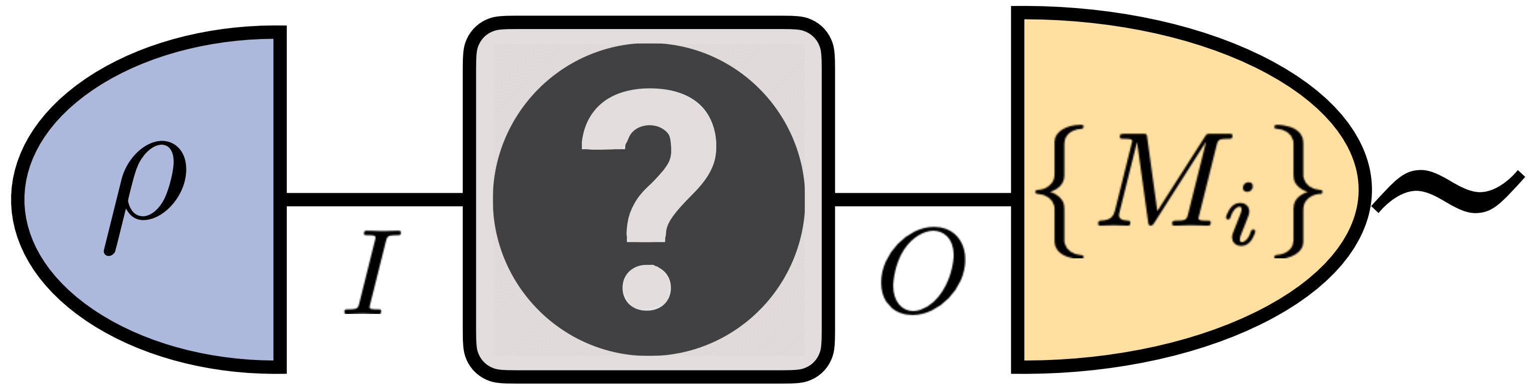
$$(T_i)^{T_O} = \rho \otimes (M_i^T)^T = \rho \otimes M_i \geq 0$$

II. Approximation: Impossibility

Channel discrimination without memory



outer approximation



$$T_i = \rho \otimes M_i^T$$

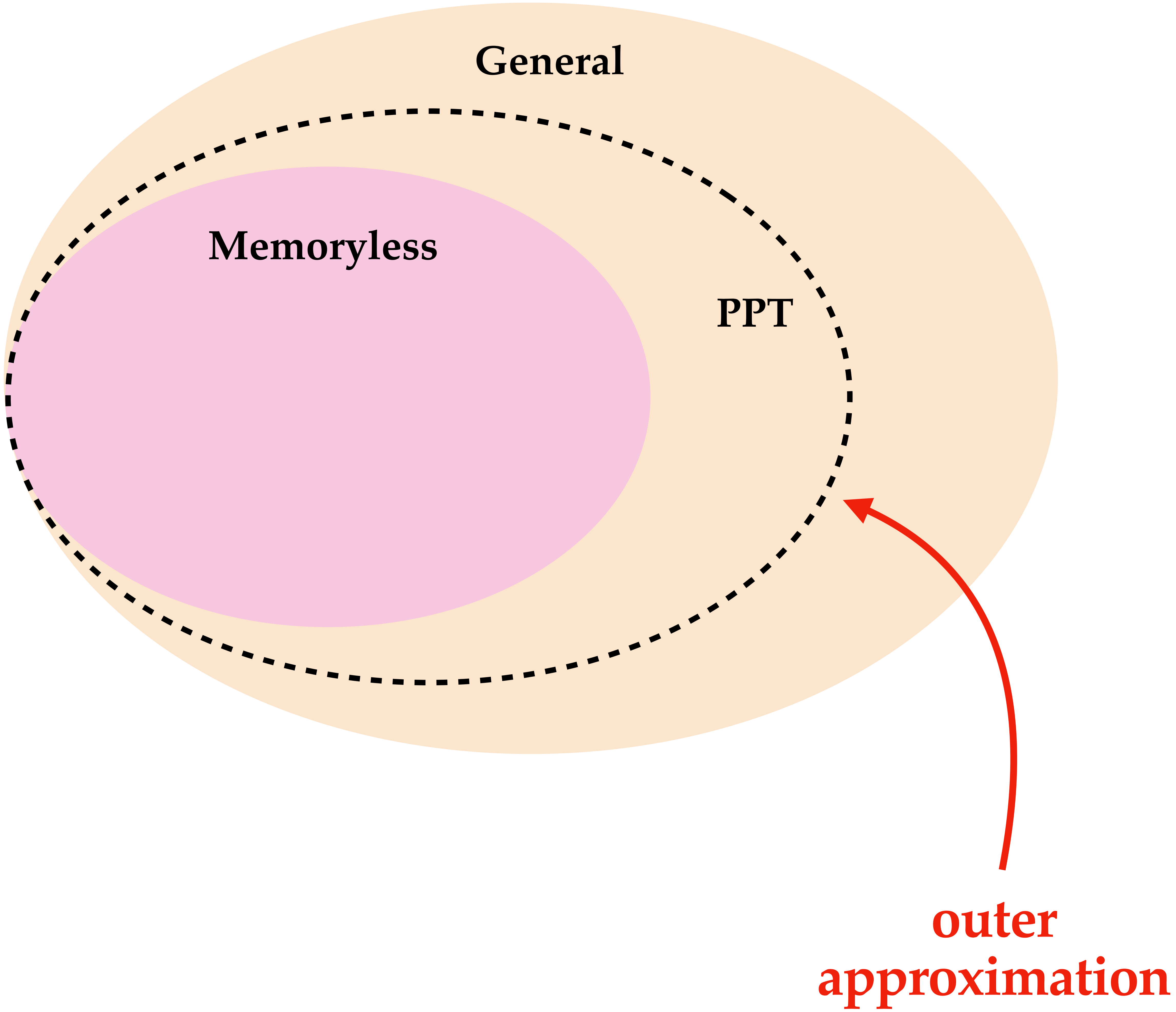
$$\Downarrow \quad \not\Uparrow$$

$$(T_i)^{T_O} = \rho \otimes (M_i^T)^T = \rho \otimes M_i \geq 0$$

positive partial transpose (PPT)

II. Approximation: Impossibility

Channel discrimination without memory



$$\text{given} \quad \{p_i, C_i\}$$

$$P_{\text{PPT}} = \max_{\{T_i\}} \sum_i p_i \operatorname{tr}\left(C_i^T T_i\right)$$

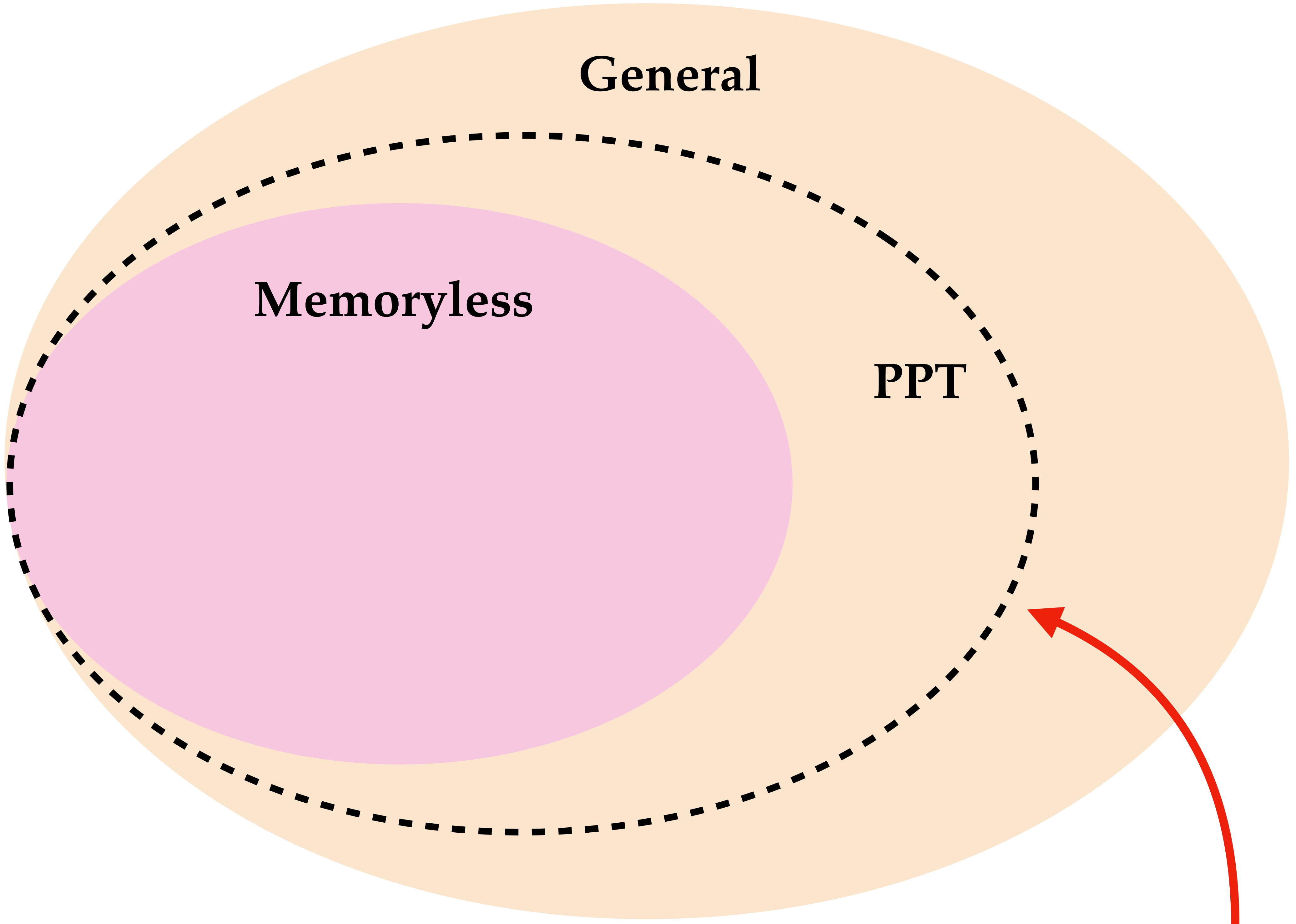
$$\text{s.t.} \quad T_i \geq 0, \quad (T_i)^{T_O} \geq 0$$

$$\operatorname{tr}\left(\sum_i T_i\right) = d_O$$

$$\sum_i T_i = \operatorname{tr}_O\left(\sum_i T_i\right) \otimes \frac{\mathbb{1}_O}{d_O}$$

II. Approximation: Impossibility

Channel discrimination without memory



$$\text{given} \quad \{p_i, C_i\}$$

$$P_{\text{PPT}} = \max_{\{T_i\}} \sum_i p_i \operatorname{tr}\left(C_i^T T_i\right)$$

$$\text{s.t.} \quad T_i \geq 0, \quad (T_i)^{T_O} \geq 0$$

$$\operatorname{tr}\left(\sum_i T_i\right) = d_O$$

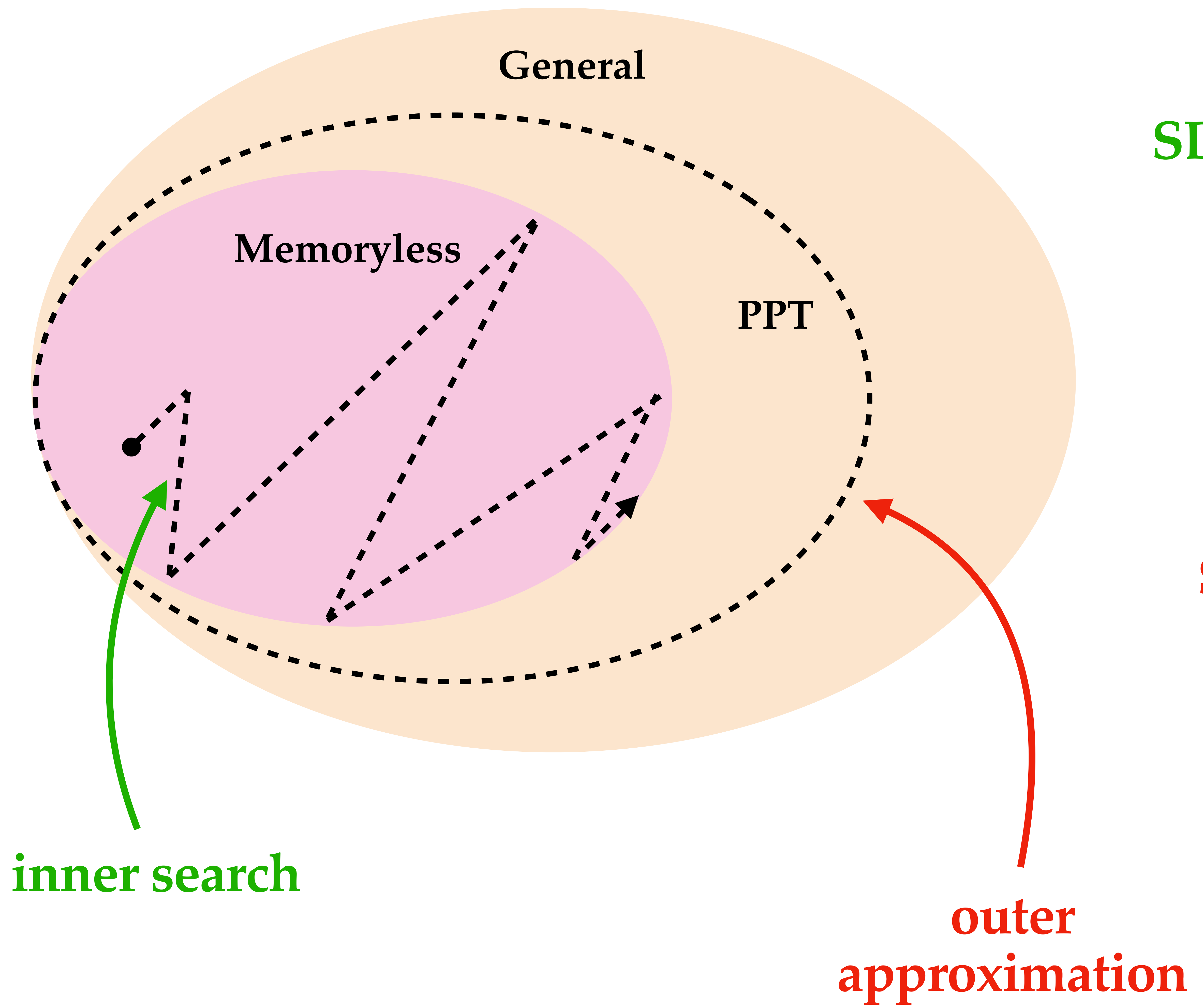
$$\sum_i T_i = \operatorname{tr}_O\left(\sum_i T_i\right) \otimes \frac{\mathbb{1}_O}{d_O}$$

$$\boxed{P_{\text{ML}} \leq P_{\text{PPT}}}$$

upper bound

II. Approximation: Impossibility

Channel discrimination without memory



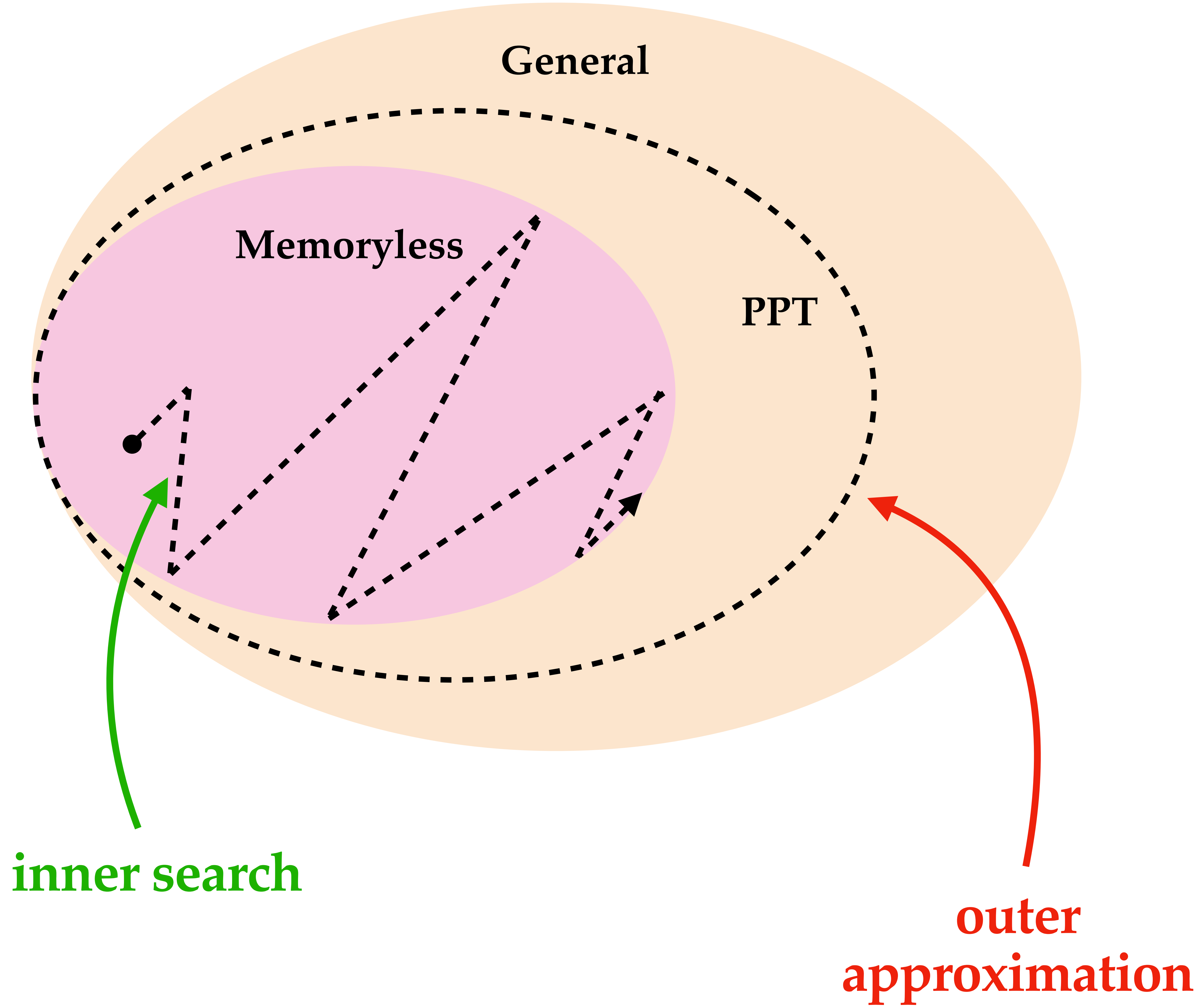
SDP lower bound: heuristic, potentially tight, but uncertain

$$\boxed{P_{\text{seesaw}} \leq P_{\text{ML}} \leq P_{\text{PPT}}}$$

SDP upper bound: deterministic but generally not tight

II. Approximation: Impossibility

Channel discrimination without memory



if $P_{\text{seesaw}} = P \implies$ optimal strategy is **memoryless**

$$\boxed{P_{\text{seesaw}} \leq P_{\text{ML}} \leq P_{\text{PPT}}} \leq P$$

if $P_{\text{PPT}} < P \implies$ optimal strategy **requires** memory

II. Approximation: Impossibility

Entanglement and the k-symmetric extension hierarchy

II. Approximation: Impossibility

Entanglement and the k-symmetric extension hierarchy

Question. How can I determine whether a given quantum state is entangled?

II. Approximation: Impossibility

Entanglement and the k-symmetric extension hierarchy

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separable quantum state:

$$\rho_{\text{sep}} = \sum_i p_i \; \rho_i^A \otimes \rho_i^B$$

II. Approximation: Impossibility

Entanglement and the k-symmetric extension hierarchy

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separable quantum state:

$$\rho_{\text{sep}} = \sum_i p_i \; \rho_i^A \otimes \rho_i^B$$

$$\begin{array}{rl} \text{given} & \rho_{AB} \\ \text{find} & \{p_i\}, \{\rho_i^A\}, \{\rho_i^B\} \\ \text{s.t.} & \rho_{AB} = \sum p_i \; \rho_i^A \otimes \rho_i^B \\ & p_i \geq 0, \; \sum_i p_i = 1 \\ & \rho_i^A \geq 0, \; \mathrm{tr}\left(\rho_i^A\right) = 1 \\ & \rho_i^B \geq 0, \; \mathrm{tr}\left(\rho_i^B\right) = 1 \end{array}$$

II. Approximation: Impossibility

Entanglement and the k-symmetric extension hierarchy

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A **seesaw** approach might prove it is separable if it **finds** a separable decomposition;
But it **can never prove** such decomposition does **not exist**.

II. Approximation: Impossibility

Entanglement and the k-symmetric extension hierarchy

Question. How can I determine whether a given quantum state is entangled?

PTT revisited:

separable quantum state:

$$\rho_{\text{sep}} = \sum_i p_i \; \rho_i^A \otimes \rho_i^B \implies (\rho_{\text{sep}})^{T_B} = \sum_i p_i \; \rho_i^A \otimes (\rho_i^B)^{T_B} \geq 0$$

II. Approximation: Impossibility

Entanglement and the k-symmetric extension hierarchy

Question. How can I determine whether a given quantum state is entangled?

$$\rho_{AB} \text{ is separable} \implies (\rho_{AB})^{T_B} \geq 0$$

II. Approximation: Impossibility

Entanglement and the k-symmetric extension hierarchy

Question. How can I determine whether a given quantum state is entangled?

$$\rho_{AB} \text{ is separable } \implies (\rho_{AB})^{T_B} \geq 0$$

$$(\rho_{AB})^{T_B} \ngeq 0 \implies \rho_{AB} \text{ is } \textbf{entangled}$$

II. Approximation: Impossibility

Entanglement and the k-symmetric extension hierarchy

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$$\rho_{AB} \text{ is separable} \implies (\rho_{AB})^{T_B} \geq 0$$

$$(\rho_{AB})^{T_B} \ngeq 0 \implies \rho_{AB} \text{ is } \textbf{entangled}$$

$$(\rho_{AB})^{T_B} \geq 0 \overset{?}{\implies} \rho_{AB} \text{ is separable}$$

What do you do when your state is PPT but you cannot find a separable decomposition via seesaw?

II. Approximation: Impossibility

Entanglement and the k-symmetric extension hierarchy

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$$\rho_{AB} \text{ is separable} \implies (\rho_{AB})^{T_B} \geq 0$$

$$(\rho_{AB})^{T_B} \ngeq 0 \implies \rho_{AB} \text{ is } \textbf{entangled}$$

$$(\rho_{AB})^{T_B} \geq 0 \;\not\Longrightarrow\; \rho_{AB} \text{ is separable}$$

What do you do when your state is PPT but you cannot find a separable decomposition via seesaw?

II. Approximation: Impossibility

Entanglement and the k-symmetric extension hierarchy

k-symmetric extension hierarchy—a.k.a the Doherty-Parrilo-Spedalieri (DPS) hierarchy

$$\rho_{\text{sep}} = \sum_i p_i \; \rho_i^A \otimes \rho_i^B$$

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symmetric $\longrightarrow$ $\rho^{(k)} = (\mathbb{1}_A \otimes U_\pi)\rho^{(k)}(\mathbb{1}_A \otimes U_\pi), \quad U_\pi \in S_k$

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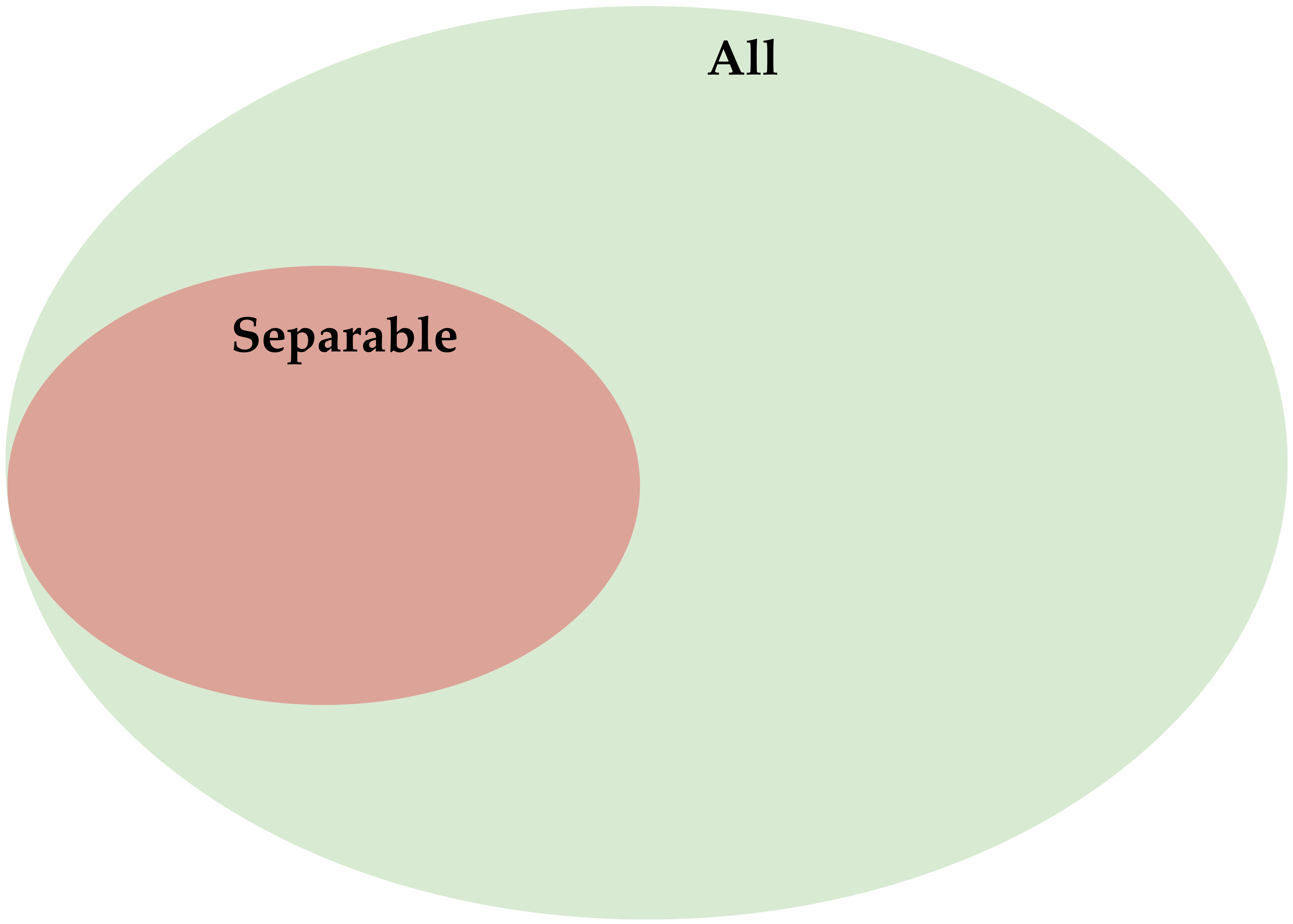

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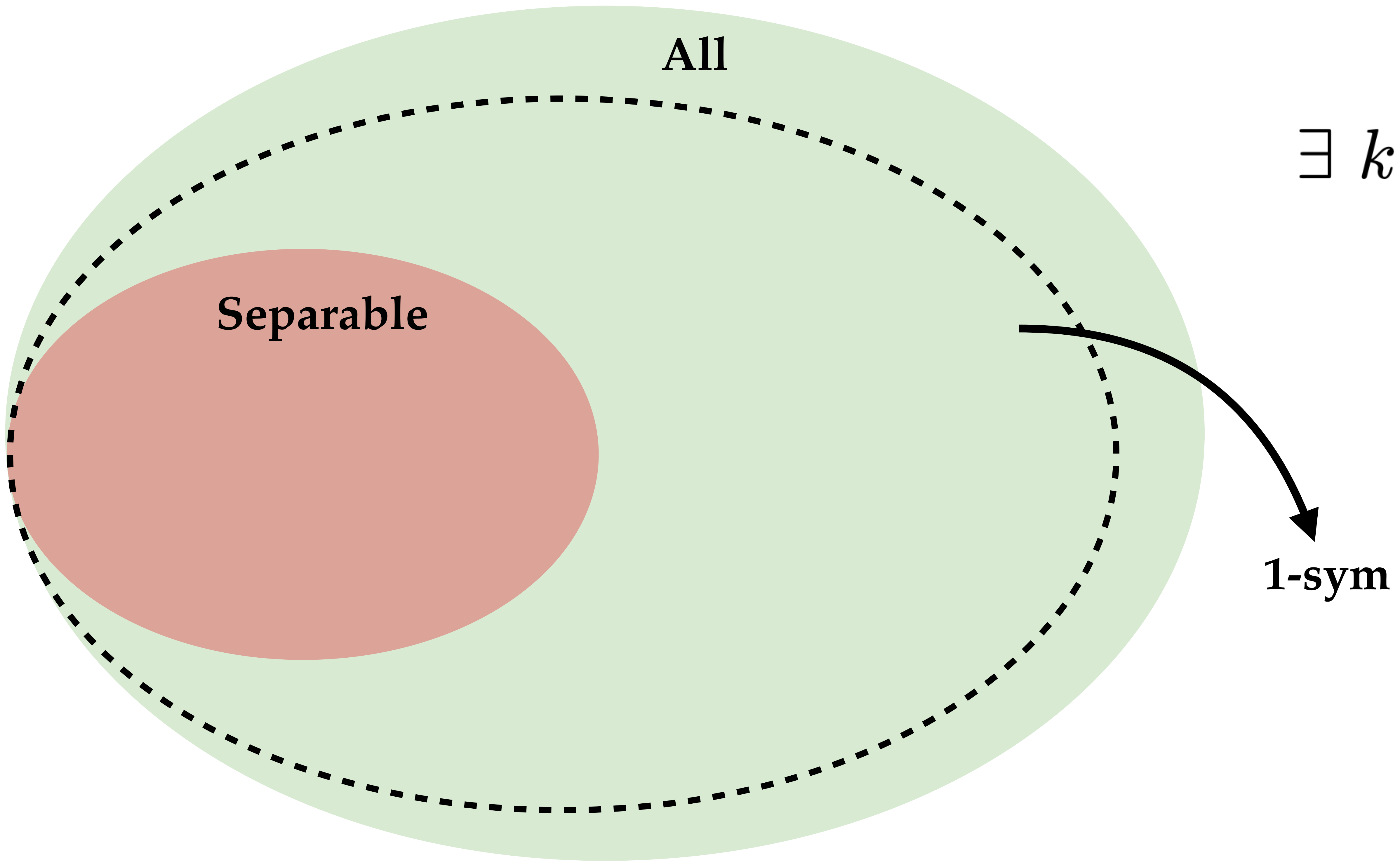

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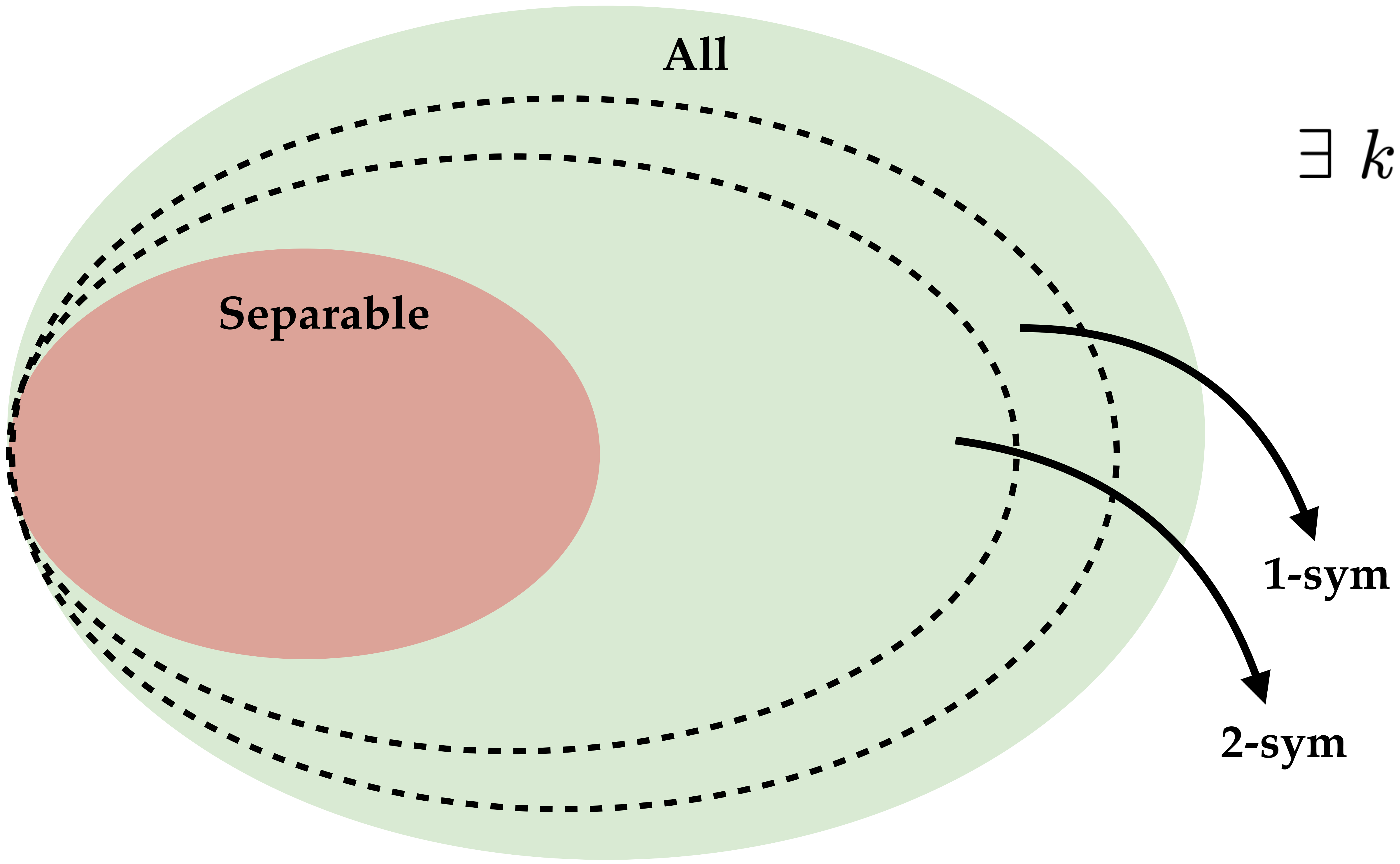

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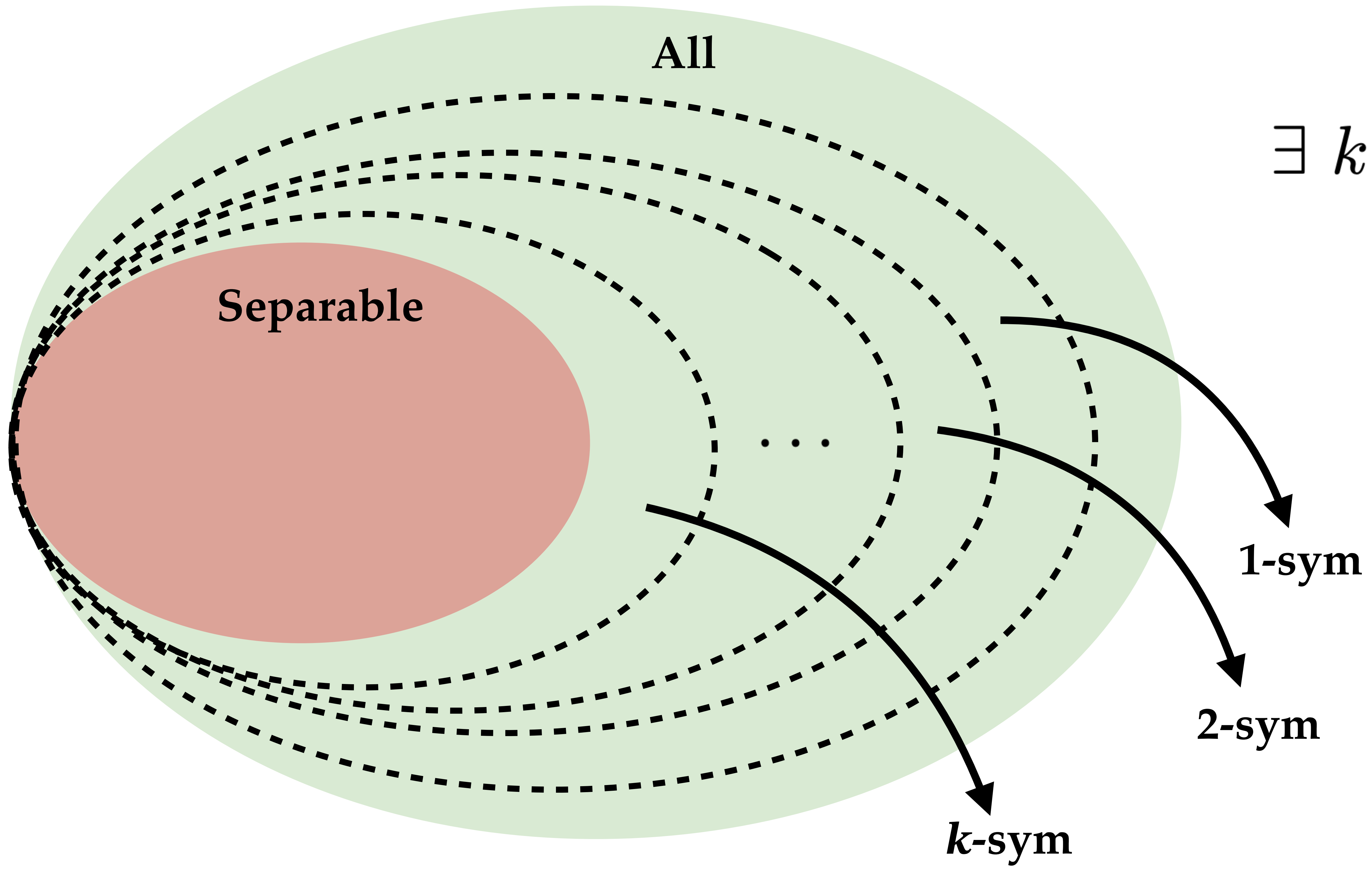

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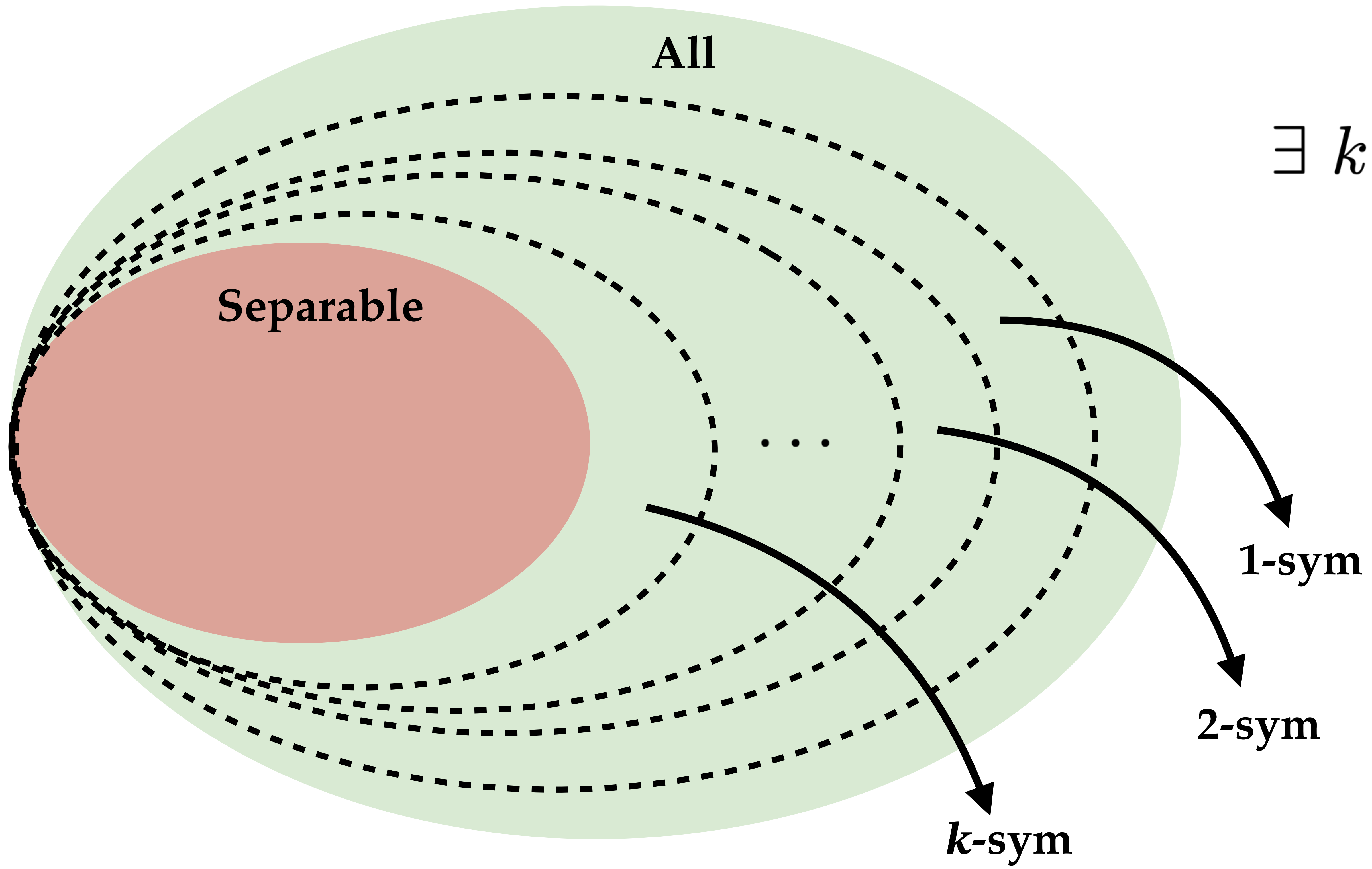

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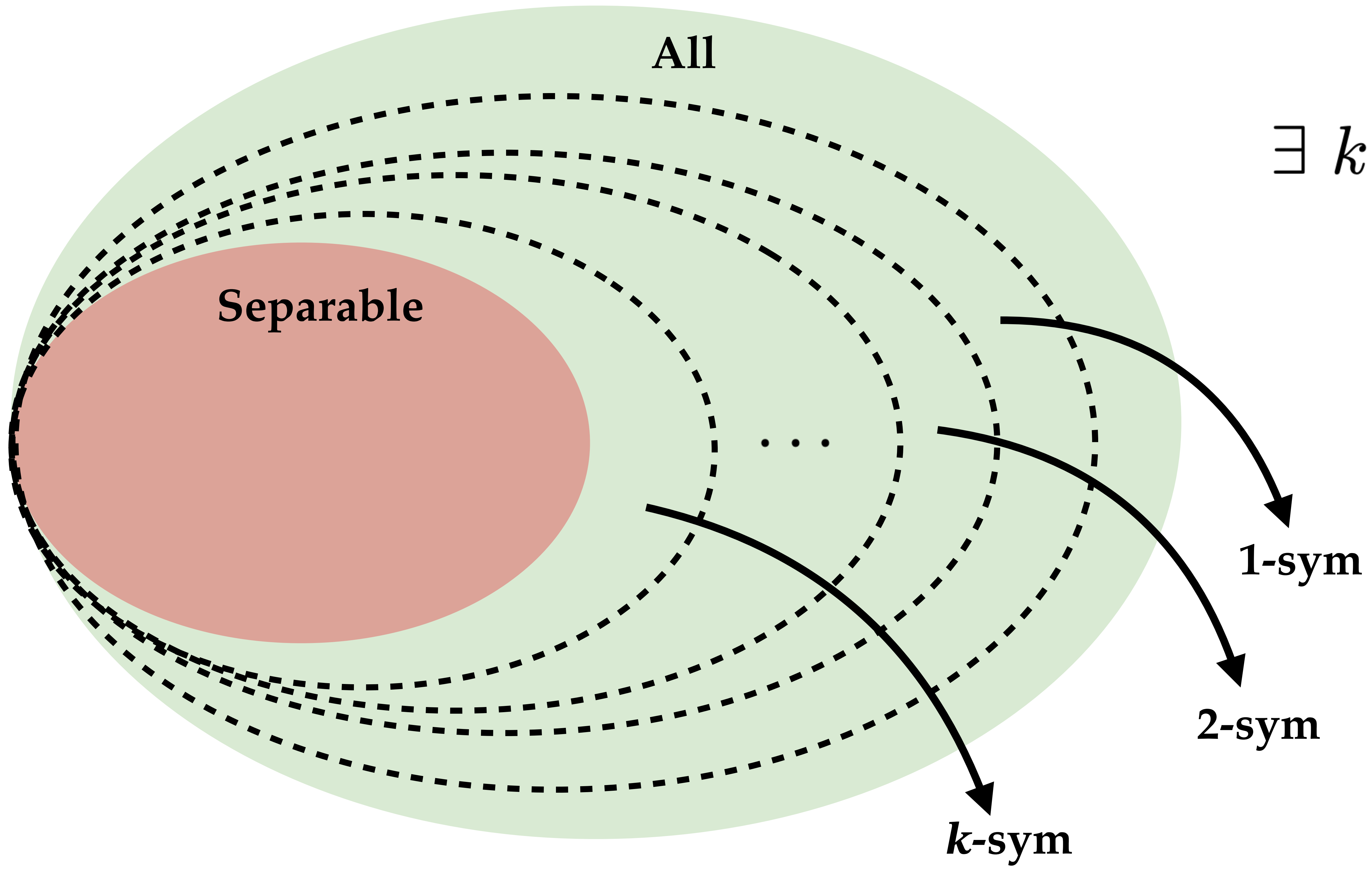

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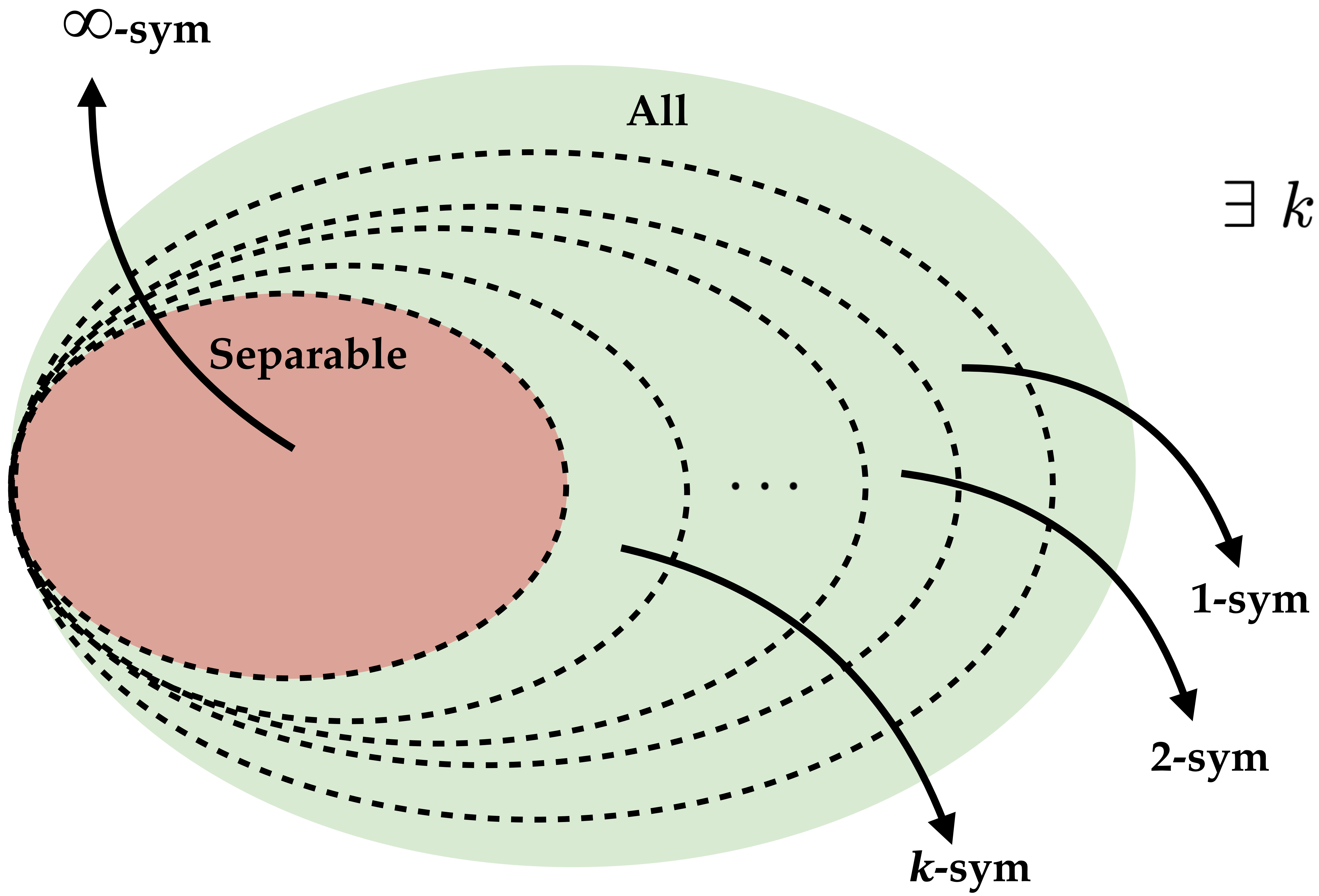

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$$\rho_{AB} \in k\text{-sym}\ \forall\ k \in \mathbb{N} \implies \rho_{AB} \text{ is separable}$$

$$\boxed{\text{separable} = \infty\text{-sym}}$$

convergence

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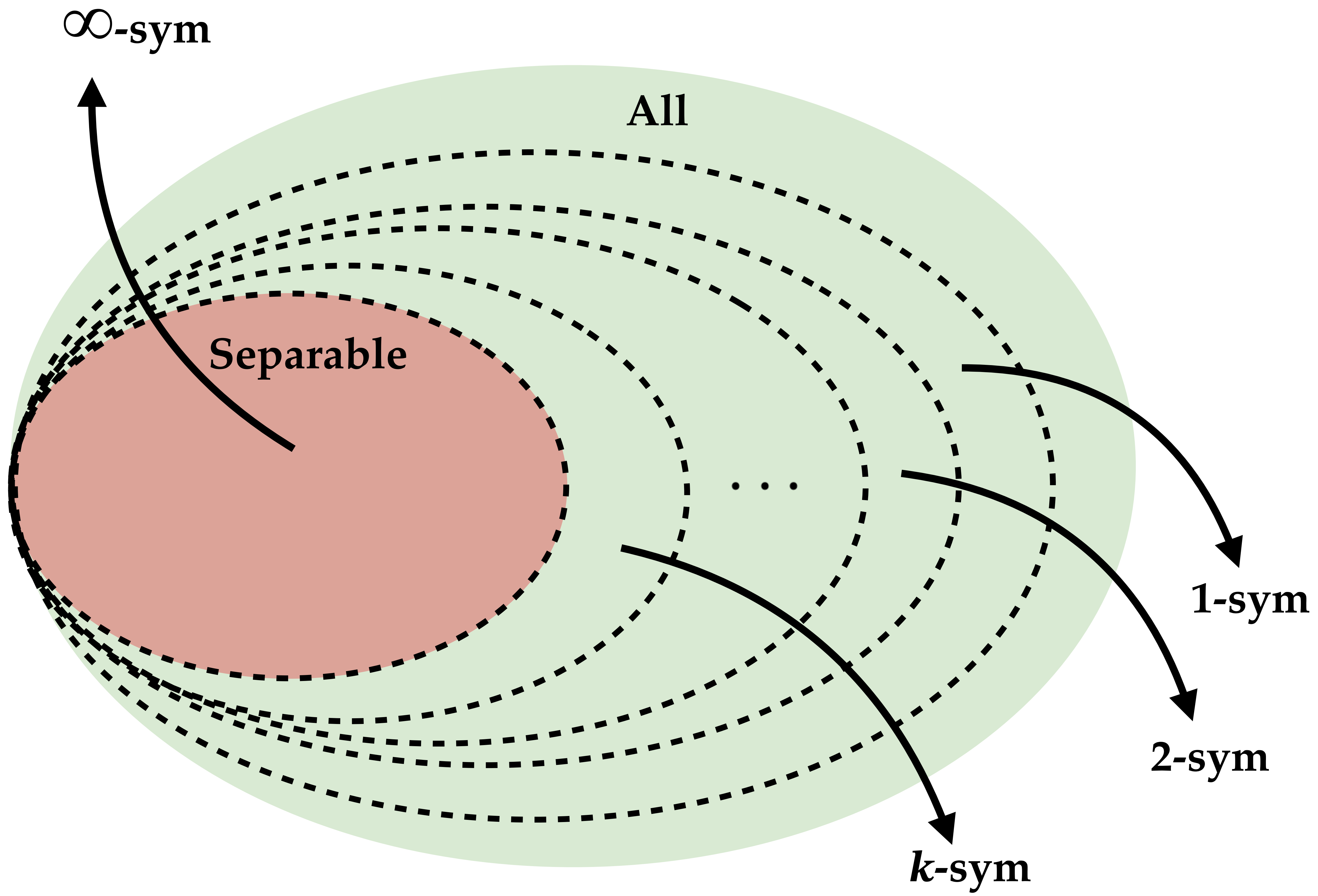


For all entangled states ρ_{AB}
$\exists k \in \mathbb{N}\ ;\ \rho_{AB} \notin k\text{-sym}$

$\text{separable} = \infty\text{-sym}$

convergence

II. Approximating: Impossibility

MAIN MESSAGE:

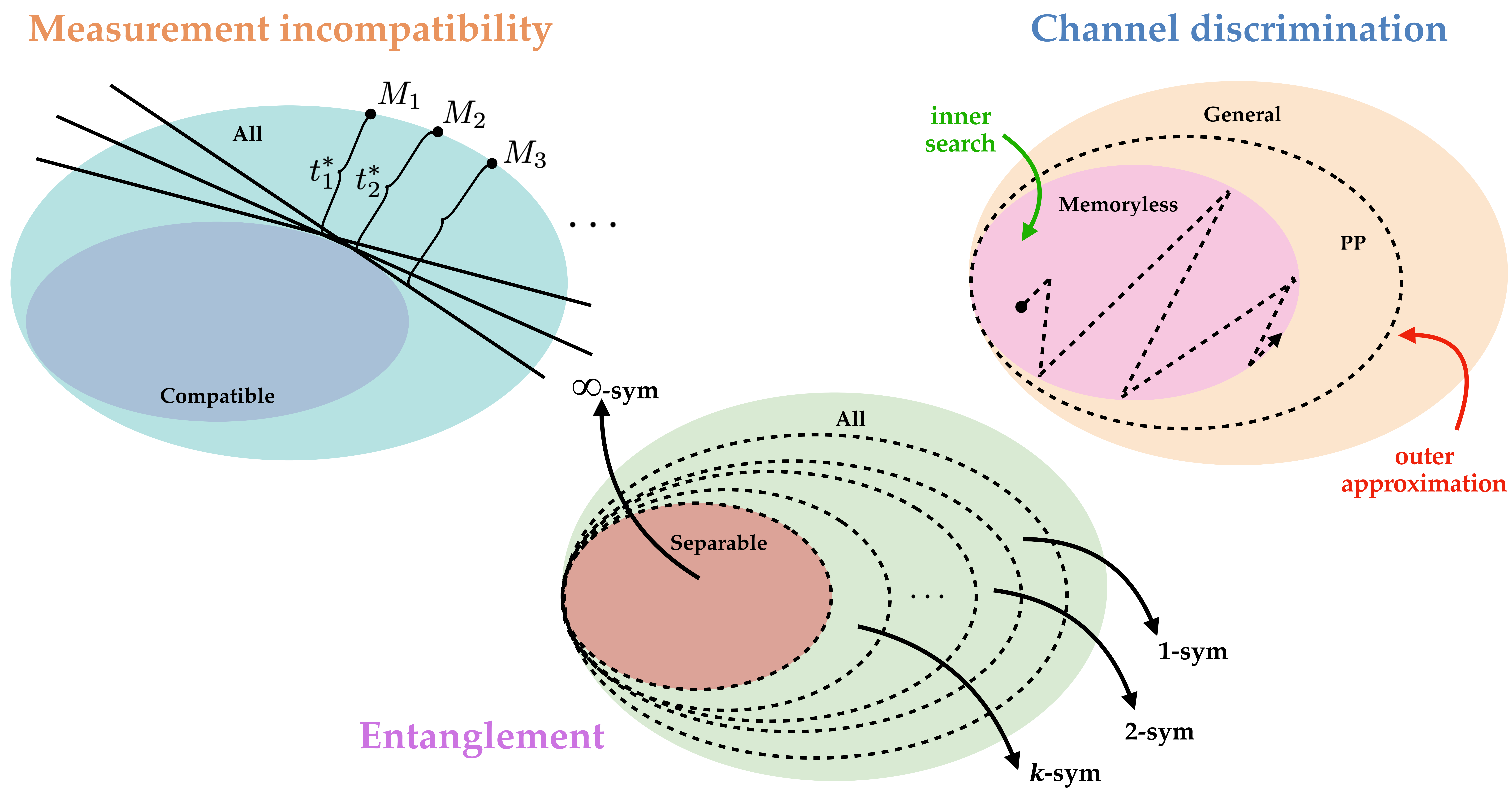
SDPs are useful tools to provide outer approximations of sets that do not accept a linear characterization.

Converging SDP hierarchies can fully characterize some very complicated problems in quantum information.

II. Approximating: Impossibility

OTHER SDP SEESAW APPLICATIONS

Summary



Measurement incompatibility
Channel discrimination
M_1
M_2
M_3
All
t_1^*
t_2^*
Compatible
. . .
inner search
General
Memoryless
PP
outer approximation
∞-sym
All
Separable
. . .
1-sym
2-sym
k-sym
Entanglement

Summary

Measurement incompatibility

- Rewriting
- Changes of variables
- Max-min -> Max-max via dual formulation
- Seesaw

Channel discrimination

- Rewriting
- Proof of linear characterization
- Seesaw
- Outer approximation via PPT

Entanglement

- Seesaw
- Outer approximation via PPT
- Outer approximation via converging hierarchy

Conclusions

- Rewriting
- Changes of variables
- Max-min -> Max-max via dual formulation
- Seesaw

Thank you!