Higher-order quantum transformations:
From quantum circuit design to indefinite causal order

Lecture II. General superchannels: Computation with indefinite causal order

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Contents

1 Introduction

Here we discuss superchannels that map several channels into one channel, that is, superchannels $\widetilde{\widetilde{S}} : [\bigotimes_{i=1}^{k}[\mathcal{L}(\mathcal{H}_{I_i}) \to \mathcal{L}(\mathcal{H}_{O_i})]] \to [\mathcal{L}(\mathcal{H}_P) \to \mathcal{L}(\mathcal{H}_F)]$ such that for any set $\{\widetilde{C}_i\}_{i=1}^{k}$ of input quantum channels, where $\widetilde{C}_i : \mathcal{L}(\mathcal{H}_{I_i}) \to \mathcal{L}(\mathcal{H}_{O_i})$, we have that

$$\widetilde{\widetilde{S}} \otimes \widetilde{\widetilde{\mathbb{1}}}(\widetilde{C}_1 \otimes \ldots \otimes \widetilde{C}_k) = \widetilde{C}' \tag{1}$$

is a quantum channel in $\widetilde{C}' : \mathcal{L}(\mathcal{H}_P) \to \mathcal{L}(\mathcal{H}_F)$.

Higher-order operations that have multiple inputs display a richer causal structure than the single-input case: some of these transformations cannot necessarily be decomposed into a quantum circuit. We will study those different classes of superchannels and understand their relation to computation with indefinite causal order. These are:

- Directly equivalent to quantum circuits
  - Parallel superchannels (non-adaptive)
  - Sequential superchannels (adaptive)
- More general than quantum circuits but with a well-defined causal order
  - Classical mixtures of quantum circuits
  - Quantum circuits with classical control of computational order
- More general than quantum circuits and with indefinite causal order
  - Quantum circuits with quantum control of computational order
  - Unitarily-extensible (purifiable) superchannels
  - General superchannels

2 Two-slot superchannels

2.1 Parallel two-slot superchannel

We start with the direct generalization of the single-slot superchannel to the two-slot case, by taking $\mathcal{H}_I \cong \mathcal{H}_{I_1} \otimes \mathcal{H}_{I_2}$ and $\mathcal{H}_O \cong \mathcal{H}_{O_1} \otimes \mathcal{H}_{O_2}$. This leads to the parallel superchannel $\widetilde{\widetilde{S}}_{\text{PAR}}$, which is a superchannel that can be implemented by a quantum circuit using a single encoder and single decoder channel. That is,

Definition 1 (Parallel two-slot superchannel)**.** *A parallel two-slot superchannel* $\widetilde{\widetilde{S}}_{\text{PAR}} : [[\mathcal{L}(\mathcal{H}_{I_1}) \to \mathcal{L}(\mathcal{H}_{O_1})] \otimes [\mathcal{L}(\mathcal{H}_{I_2}) \to \mathcal{L}(\mathcal{H}_{O_2})]] \to [\mathcal{L}(\mathcal{H}_P) \to \mathcal{L}(\mathcal{H}_F)]$ *that transforms two quantum channels* $\widetilde{C}_1 : \mathcal{L}(\mathcal{H}_{I_1}) \to \mathcal{L}(\mathcal{H}_{O_1})$ *and* $\widetilde{C}_2 : \mathcal{L}(\mathcal{H}_{I_2}) \to \mathcal{L}(\mathcal{H}_{O_2})$ *into another quantum channel* $\widetilde{C}' : \mathcal{L}(\mathcal{H}_P) \to \mathcal{L}(\mathcal{H}_F)$ *in such a way that it acts on channels* $\widetilde{A}$ *and* $\widetilde{B}$ *in a parallel manner, is a superchannel that can be realized by a quantum circuit composed of the following channels: an encoder quantum channel* $\widetilde{E} : \mathcal{L}(\mathcal{H}_P) \to \mathcal{L}(\mathcal{H}_{I_1} \otimes \mathcal{H}_{I_2} \otimes \mathcal{H}_M)$ *and a decoder quantum channel* $\widetilde{D} : \mathcal{L}(\mathcal{H}_M \otimes \mathcal{H}_{O_1} \otimes \mathcal{H}_{O_2}) \to \mathcal{L}(\mathcal{H}_F)$*, in such a way that*

$$\widetilde{C}' = \widetilde{\widetilde{S}}_{\text{PAR}}(\widetilde{C}_1 \otimes \widetilde{C}_2) = \widetilde{D} \circ (\widetilde{\mathbb{1}} \otimes \widetilde{C}_1 \otimes \widetilde{C}_2) \circ \widetilde{E}, \tag{2}$$

is a quantum channel for all input channels $\widetilde{C}_1$ *and* $\widetilde{C}_2$*.*

In the Choi representation, this is equivalent to saying that

$$S_{\text{PAR}} = E * D. \tag{3}$$

Hence, the characterization of S_{PAR} in terms of positivity constraints and linear constraints, as well as its realization in terms of E and D is analogous. We repeat it below for sake of completeness.

Proposition 1 (Characterization of parallel two-slot superchannels)**.** *An operator* $S_{\text{PAR}} \in \mathcal{L}(\mathcal{H}_P \otimes \mathcal{H}_{I_1} \otimes \mathcal{H}_{O_1} \otimes \mathcal{H}_{I_2} \otimes \mathcal{H}_{O_2} \otimes \mathcal{H}_F)$ *corresponds to the Choi operator of a parallel two-slot superchannel, i.e., there exist Choi operators of quantum channels* $\widetilde{E} : \mathcal{L}(\mathcal{H}_P) \to \mathcal{L}(\mathcal{H}_{I_1} \otimes \mathcal{H}_{I_2} \otimes \mathcal{H}_M)$ *and* $\widetilde{D} : \mathcal{L}(\mathcal{H}_M \otimes \mathcal{H}_{O_1} \otimes \mathcal{H}_{O_2}) \to \mathcal{L}(\mathcal{H}_F)$ *such that* $S_{\text{PAR}} = E * D$*, if and only if*

$$S_{\text{PAR}} \geq 0 \tag{4}$$
$$\operatorname{tr}(S_{\text{PAR}}) = d_P d_{O_1} d_{O_2} \tag{5}$$
$${}_F S_{\text{PAR}} = {}_{O_1 O_2 F} S_{\text{PAR}} \tag{6}$$
$${}_{I_1 I_2 O_1 O_2 F} S_{\text{PAR}} = {}_{P I_1 I_2 O_1 O_2 F} S_{\text{PAR}}. \tag{7}$$

Proof. By direct calculation, it can be checked that

$$S_{\text{PAR}} = E * D = \operatorname{tr}_{\text{aux}}\left[(E^{T_{\text{aux}}} \otimes \mathbb{1}_{O_1 O_2 F})(\mathbb{1}_{P I_1 I_2} \otimes D)\right] \tag{8}$$

satisfies Eqs. (4)–(7). To prove that, for every S_{PAR} that satisfies Eqs. (4)–(7), there exists E and D, such that $S_{\text{PAR}} = E * D$, take the following ansatz: Let $\sigma := \operatorname{tr}_{O_1 O_2 F}(S_{\text{PAR}})/(d_{O_1} d_{O_2})$ be embedded in $\mathcal{L}(\mathcal{H}_M)$, where $\mathcal{H}_M \cong \mathcal{H}_P \otimes \mathcal{H}_{I_1} \otimes \mathcal{H}_{I_2}$. Then,

$$E := \left(\mathbb{1}_{P I_1 I_2} \otimes \sqrt{\sigma}^T\right) \sum_{ij} |ii\rangle\langle jj| \left(\mathbb{1}_{P I_1 I_2} \otimes \sqrt{\sigma}^T\right) \tag{9}$$
$$D := \left(\sqrt{\sigma}^{-1} \otimes \mathbb{1}_{O_1 O_2 F}\right) S_{\text{PAR}} \left(\sqrt{\sigma}^{-1} \otimes \mathbb{1}_{O_1 O_2 F}\right). \tag{10}$$

By direct calculation, it can be checked that E and D are Choi operators of quantum channels and that $E * D = S_{\text{PAR}}$, concluding the proof. □

Equations (6)–(7) can be conveniently expressed as a single projector, the projector $\mathbb{P}_{\text{PAR}}$ onto the subspace of parallel two-slot superchannels, according to

$$S_{\text{PAR}} \geq 0 \tag{11}$$
$$\operatorname{tr}(S_{\text{PAR}}) = d_P d_{O_1} d_{O_2} \tag{12}$$
$$S_{\text{PAR}} = \mathbb{P}_{\text{PAR}}(S_{\text{PAR}}) := S_{\text{PAR}} - {}_F S_{\text{PAR}} + {}_{O_1O_2F} S_{\text{PAR}} - {}_{I_1I_2O_1O_2F} S_{\text{PAR}} + {}_{PI_1I_2O_1O_2F} S_{\text{PAR}}. \tag{13}$$

2.2 Sequential two-slot superchannel

A sequential two-slot superchannel is a superchannel that can be implemented by a quantum circuit that applies one of the input channels before the other, in a fixed, predetermined order, allowing for arbitrary intermediary quantum operations.

That is,

Definition 2 (Sequential two-slot superchannel). *A sequential two-slot superchannel $\widetilde{\widetilde{S}}_{\text{SEQ}} : [[\mathcal{L}(\mathcal{H}_{I_1}) \to \mathcal{L}(\mathcal{H}_{O_1})] \otimes [\mathcal{L}(\mathcal{H}_{I_2}) \to \mathcal{L}(\mathcal{H}_{O_2})]] \to [\mathcal{L}(\mathcal{H}_P) \to \mathcal{L}(\mathcal{H}_F)]$ that transforms two quantum channels $\widetilde{C}_1 : \mathcal{L}(\mathcal{H}_{I_1}) \to \mathcal{L}(\mathcal{H}_{O_1})$ and $\widetilde{C}_2 : \mathcal{L}(\mathcal{H}_{I_2}) \to \mathcal{L}(\mathcal{H}_{O_2})$ into another quantum channel $\widetilde{C} : \mathcal{L}(\mathcal{H}_P) \to \mathcal{L}(\mathcal{H}_F)$ in such a way that it acts first on channel $\widetilde{C}_1$ and then on channel $\widetilde{C}'$ in a sequential manner, is a superchannel that can be realized by a quantum circuit composed of the following channels: a first encoder quantum channel $\widetilde{E}_1 : \mathcal{L}(\mathcal{H}_P) \to \mathcal{L}(\mathcal{H}_{I_1} \otimes \mathcal{H}_{M_1})$, a second encoder quantum channel $\widetilde{E}_2 : \mathcal{L}(\mathcal{H}_{M_1} \otimes \mathcal{H}_{O_1}) \to \mathcal{L}(\mathcal{H}_{I_2} \otimes \mathcal{H}_{M_2})$, and a decoder quantum channel $\widetilde{D} : \mathcal{L}(\mathcal{H}_{M_2} \otimes \mathcal{H}_{O_2}) \to \mathcal{L}(\mathcal{H}_F)$, in such a way that*

$$\widetilde{C}' = \widetilde{\widetilde{S}}_{\text{SEQ}}(\widetilde{C}_1 \otimes \widetilde{C}_2) = \widetilde{D} \circ (\widetilde{\mathbb{1}} \otimes \widetilde{C}_2) \circ \widetilde{E}_2 \circ (\widetilde{\mathbb{1}} \otimes \widetilde{C}_1) \circ \widetilde{E}_1, \tag{14}$$

is a quantum channel for all input channels $\widetilde{C}_1$ and $\widetilde{C}_2$.

In the Choi representation, this is equivalent to saying that

$$S_{\text{SEQ}} = E_1 * E_2 * D. \tag{15}$$

We know show simple conditions on the operator S_{SEQ} that are equivalent to saying it it admits a decomposition of the form $S_{\text{SEQ}} = E_1 * E_2 * D$, that is, that it is the Choi operators of a sequential two-slot superchannel.

Proposition 2 (Characterization of sequential two-slot superchannels). *An operator $S_{\text{SEQ}} \in \mathcal{L}(\mathcal{H}_P \otimes \mathcal{H}_{I_1} \otimes \mathcal{H}_{O_1} \otimes \mathcal{H}_{I_2} \otimes \mathcal{H}_{O_2} \otimes \mathcal{H}_F)$ corresponds to the Choi operator of a sequential two-slot superchannel, i.e., there exist Choi operators of quantum channels $E_1 \in \mathcal{L}(\mathcal{H}_P \otimes \mathcal{H}_{I_1} \otimes \mathcal{H}_{M_1})$, $E_2 \in \mathcal{L}(\mathcal{H}_{M_1} \otimes \mathcal{H}_{O_1} \otimes \mathcal{H}_{I_2} \otimes \mathcal{H}_{M_2})$ and $D \in \mathcal{L}(\mathcal{H}_{M_2} \otimes \mathcal{H}_{O_2} \otimes \mathcal{H}_F)$ such that $S_{\text{SEQ}} = E_1 * E_2 * D$, if and only if*

$$S_{\text{SEQ}} \geq 0 \tag{16}$$
$$\operatorname{tr}(S_{\text{SEQ}}) = d_P d_{O_1} d_{O_2} \tag{17}$$
$${}_F S_{\text{SEQ}} = {}_{O_2F} S_{\text{SEQ}} \tag{18}$$
$${}_{I_2O_2F} S_{\text{SEQ}} = {}_{O_1I_2O_2F} S_{\text{SEQ}} \tag{19}$$
$${}_{I_1O_1I_2O_2F} S_{\text{SEQ}} = {}_{PI_1O_1I_2O_2F} S_{\text{SEQ}}. \tag{20}$$

Proof. By direct calculation, it can be checked that

$$S_{\text{SEQ}} = E_1 * E_2 * D \tag{21}$$

satisfies Eqs. (16)–(20). To prove that, for every S_{SEQ} that satisfies Eqs. (16)–(20), there exists E_1, E_2, and D, such that $S_{\text{SEQ}} = E_1 * E_2 * D$, take the following ansatz: Let $\sigma := \operatorname{tr}_{O_1I_2O_2F}(S_{\text{SEQ}})$ be embedded in $\mathcal{L}(\mathcal{H}_{M_1})$, where $\mathcal{H}_{M_1} \cong \mathcal{H}_P \otimes \mathcal{H}_{I_1}$, and let $\omega := \mathrm{tr}_{O_2F}(S_{\mathrm{SEQ}})/d_{O_2}$ be embedded in $\mathcal{L}(\mathcal{H}_{M_2})$, where $\mathcal{H}_{M_2} \cong \mathcal{H}_P \otimes \mathcal{H}_{I_1} \otimes \mathcal{H}_{O_1} \otimes \mathcal{H}_{I_2}$. Then,

$$E_1 := \left(\mathbb{1}_{PI_1} \otimes \sqrt{\sigma}^T\right) \sum_{ij} |ii\rangle\langle jj| \left(\mathbb{1}_{PI_1} \otimes \sqrt{\sigma}^T\right) \tag{22}$$

$$E_2 := \left(\sqrt{\sigma}^{-1} \otimes \mathbb{1}_{O_1I_2M_2}\right) \left(\mathbb{1}_{M_2} \otimes \sqrt{\omega}^T\right) \sum_{ij} |ii\rangle\langle jj| \left(\sqrt{\sigma}^{-1} \otimes \mathbb{1}_{O_1I_2M_2}\right) \left(\mathbb{1}_{M_2} \otimes \sqrt{\omega}^T\right) \tag{23}$$

$$D := \left(\sqrt{\omega}^{-1} \otimes \mathbb{1}_{O_2F}\right) S_{\mathrm{SEQ}} \left(\sqrt{\omega}^{-1} \otimes \mathbb{1}_{O_2F}\right). \tag{24}$$

By direct calculation, it can be checked that E_1, E_2, and D are Choi operators of quantum channels and that $E_1 * E_2 * D = S_{\mathrm{SEQ}}$, concluding the proof. □

Equations (18)–(20) can be conveniently expressed with a single projector, the projector $\mathbb{P}_{\mathrm{SEQ}}$ onto the subspace of sequential two-slot superchannels, according to

$$S_{\mathrm{SEQ}} \geq 0 \tag{25}$$

$$\mathrm{tr}(S_{\mathrm{SEQ}}) = d_P d_{O_1} d_{O_2} \tag{26}$$

$$\begin{aligned} S_{\mathrm{SEQ}} = \mathbb{P}_{\mathrm{SEQ}}(S_{\mathrm{SEQ}}) := S_{\mathrm{SEQ}} - {}_F S_{\mathrm{SEQ}} + {}_{O_2F} S_{\mathrm{SEQ}} - {}_{I_2O_2F} S_{\mathrm{SEQ}} + {}_{O_1I_2O_2F} S_{\mathrm{SEQ}} + \\ - {}_{I_1O_1I_2O_2F} S_{\mathrm{SEQ}} + {}_{PI_1O_1I_2O_2F} S_{\mathrm{SEQ}}. \end{aligned} \tag{27}$$

Finally, note that unlike the case of general superchannels, the set of parallel superchannels is not invariant under the permutation of the systems $I_1 \leftrightarrow I_2$ and $O_1 \leftrightarrow O_2$. In fact, the superchannels S_{SEQ} can be thought of as specifically "$S_{1\prec 2}$", that is, as superchannels that act first on channel C_1 and then on channel C_2. By permuting $I_1 \leftarrow I_2$ and $O_1 \leftarrow O_2$, we can define explicitly sequential superchannels $S_{2\prec 1}$ that act first on channel C_2 and then on channel C_1. The intersection of the set of superchannels that simultaneously satisfy both the conditions of $S_{1\prec 2}$ and $S_{2\prec 1}$ is precisely the set of parallel superchannels.

2.3 General two-slot superchannels

The previous to definitions of sets of superchannels were motivated by their quantum-circuit realization: parallel superchannels are those that can be implemented by a quantum circuit with open slots that calls the input quantum channels in parallel, and sequential superchannels are those that can be implemented by a quantum circuit with open slots that calls the input quantum channels sequentially.

From a higher-order transformations point of view, however, an interesting question is what is the most general (multi-)linear transformation that maps any set of input channels into a quantum channel, even when acting only on part of the input channels. That is, what is the most general completely CPTP preserving transformation that has many inputs?

In the two-slot case, we formalize this definition below.

Definition 3 (General two-slot superchannel)**.** *A general two-slot superchannel* $\widetilde{\widetilde{S}}_{\mathrm{GEN}} : [[\mathcal{L}(\mathcal{H}_{I_1}) \to \mathcal{L}(\mathcal{H}_{O_1})] \otimes [\mathcal{L}(\mathcal{H}_{I_2}) \to \mathcal{L}(\mathcal{H}_{O_2})]] \to [\mathcal{L}(\mathcal{H}_P) \to \mathcal{L}(\mathcal{H}_F)]$ *is the most general bilinear map that transforms any two independent channels,* $\widetilde{C}_{\mathrm{ext}_1} : \mathcal{L}(\mathcal{H}_{I_1} \otimes \mathcal{H}_{I_1'}) \to \mathcal{L}(\mathcal{H}_{O_1} \otimes \mathcal{H}_{O_1'})$ *and* $\widetilde{C}_{\mathrm{ext}_2} : \mathcal{L}(\mathcal{H}_{I_2} \otimes \mathcal{H}_{I_2'}) \to \mathcal{L}(\mathcal{H}_{O_2} \otimes \mathcal{H}_{O_2'})$*, into another channel* $\widetilde{C}'_{\mathrm{ext}} : \mathcal{L}(\mathcal{H}_{I_1'} \otimes \mathcal{H}_P \otimes \mathcal{H}_{I_2'}) \to \mathcal{L}(\mathcal{H}_{O_1'} \otimes \mathcal{H}_F \otimes \mathcal{H}_{O_2'})$*, even when acting only on part of the input channels, according to*

$$\widetilde{\widetilde{S}}_{\mathrm{GEN}} \otimes \widetilde{\widetilde{\mathbb{1}}}(\widetilde{C}_{\mathrm{ext}_1} \otimes \widetilde{C}_{\mathrm{ext}_2}) = \widetilde{C}'_{\mathrm{ext}}. \tag{28}$$

The two-slot superchannel, just as in the one-slot case, is therefore defined as a **completely CPTP preserving bilinear map**. In the Choi representation, the above equation becomes

$$C_{\text{ext}} = \widetilde{S}_{\text{GEN}} \otimes \widetilde{\mathbb{1}}(C_{\text{ext}_1} \otimes C_{\text{ext}_2}) = S_{\text{GEN}} * (C_{\text{ext}_1} \otimes C_{\text{ext}_2}) = S_{\text{GEN}} * C_{\text{ext}_1} * C_{\text{ext}_2} \tag{29}$$

$$= \text{tr}_{I_1O_1I_2O_2}[(C_{\text{ext}_1}^{T_{I_1O_1}} \otimes C_{\text{ext}_2}^{T_{I_2O_2}} \otimes \mathbb{1}_{PF})(\mathbb{1}_{I_1'O_1'I_2'O_2'} \otimes S_{\text{GEN}})]. \tag{30}$$

In this case once again, we have that complete CP preservation is different from simply CP preservation and that, contrarily, complete TP preservation is equivalent to TP preservation. Hence, when studying the implications of the complete TP preservation condition on the Choi operator S_{GEN}, it suffices to analyze the case where all auxiliary systems $\mathcal{H}_{I_1'} \cong \mathcal{H}_{O_1'} \cong \mathcal{H}_{I_2'} \cong \mathcal{H}_{O_2'}$ have dimension 1, and the above equation reduces to

$$C = \widetilde{S}_{\text{GEN}}(C_1 \otimes C_2) = S_{\text{GEN}} * (C_1 \otimes C_2) = S_{\text{GEN}} * C_1 * C_2 \tag{31}$$

$$= \text{tr}_{I_1O_1I_2O_2}[(C_1^T \otimes C_2^T \otimes \mathbb{1}_{PF})\, S_{\text{GEN}}]. \tag{32}$$

Proposition 3 (Characterization of general two-slot superchannels)**.** *A linear operator $S_{\text{GEN}} \in \mathcal{L}(\mathcal{H}_P \otimes \mathcal{H}_{I_1} \otimes \mathcal{H}_{O_1} \otimes \mathcal{H}_{I_2} \otimes \mathcal{H}_{O_2} \otimes \mathcal{H}_F)$ is the Choi operator of a general two-slot superchannel, i.e., is a linear operator such that, for every Choi operator of an extended quantum channels $C_{\text{ext}_1} \in \mathcal{L}(\mathcal{H}_{I_1} \otimes \mathcal{H}_{I_1'} \otimes \mathcal{H}_{O_1} \otimes \mathcal{H}_{O_1'})$ and $C_{\text{ext}_2} \in \mathcal{L}(\mathcal{H}_{I_2} \otimes \mathcal{H}_{I_2'} \otimes \mathcal{H}_{O_2} \otimes \mathcal{H}_{O_2'})$, one has that $S_{\text{GEN}} * C_{\text{ext}_1} * C_{\text{ext}_2} = C'_{\text{ext}} \in \mathcal{L}(\mathcal{H}_{I_1'} \otimes \mathcal{H}_P \otimes \mathcal{H}_{I_2'} \otimes \mathcal{H}_{O_1'} \otimes \mathcal{H}_F \otimes \mathcal{H}_{O_2'})$ is a valid extended quantum channel, if and only if*

$$S_{\text{GEN}} \geq 0 \tag{33}$$

$$\text{tr}(S_{\text{GEN}}) = d_P d_{O_1} d_{O_2} \tag{34}$$

$${}_{I_1O_1F}S_{\text{GEN}} = {}_{I_1O_1O_2F}S_{\text{GEN}} \tag{35}$$

$${}_{I_2O_2F}S_{\text{GEN}} = {}_{O_1I_2O_2F}S_{\text{GEN}} \tag{36}$$

$${}_{F}S_{\text{GEN}} = {}_{O_1F}S_{\text{GEN}} + {}_{O_2F}S_{\text{GEN}} - {}_{O_1O_2F}S_{\text{GEN}} \tag{37}$$

$${}_{I_1O_1I_2I_2F}S_{\text{GEN}} = {}_{PI_1O_1I_2I_2F}S_{\text{GEN}}. \tag{38}$$

Proof. For the proof of these necessary and sufficient conditions on the Choi operator of a general two-slot superchannel S_{GEN}, see Appendix C of Ref. [1]. See also Appendix B of Ref. [2] for the proof of the case where $d_P = d_F = 1$. □

The conditions in Eqs. (33)–(38) for an operator S_{GEN} to be a valid superchannel can also be equivalently expressed as

$$S_{\text{GEN}} \geq 0 \tag{39}$$

$$\text{tr}(S_{\text{GEN}}) = d_P d_{O_1} d_{O_2} \tag{40}$$

$$\begin{aligned} S_{\text{GEN}} = \mathbb{P}_{\text{GEN}}(S_{\text{GEN}}) := {} & S_{\text{GEN}} - {}_{F}S_{\text{GEN}} + {}_{O_1F}S_{\text{GEN}} + {}_{O_2F}S_{\text{GEN}} - {}_{O_1O_2F}S_{\text{GEN}} + \\ & - {}_{I_1O_1F}S_{\text{GEN}} + {}_{I_1O_1O_2F}S_{\text{GEN}} - {}_{I_2O_2F}S_{\text{GEN}} + {}_{O_1I_2O_2F}S_{\text{GEN}} + \\ & - {}_{I_1O_1I_2O_2F}S_{\text{GEN}} + {}_{PI_1O_1I_2O_2F}S_{\text{GEN}}, \end{aligned} \tag{41}$$

where $\mathbb{P}_{\text{GEN}}$ is the projector onto the subspace of general two-slot superchannels.

Contrarily to the 1-slot case, it is known that *not* every general two-slot superchannel can be realized by a quantum circuit. The paradigmatic example is the quantum switch [3], which is interpreted as displaying an indefinite causal order.

3 The quantum switch

The quantum switch, first defined in Ref. [3], is a superchannel whose order of application of the input channels is conditioned on the state of a quantum control system. One of its features is that it maps all unitary channels into a unitary channel.

We begin by defining the quantum switch as a higher-order transformation $\widetilde{\widetilde{S}}_{\mathrm{QS}} : [[\mathcal{L}(\mathcal{H}_{A_I}) \to \mathcal{L}(\mathcal{H}_{A_O})] \otimes [\mathcal{L}(\mathcal{H}_{B_I}) \to \mathcal{L}(\mathcal{H}_{B_O})]] \to [\mathcal{L}(\mathcal{H}_{P_c} \otimes \mathcal{H}_{P_t}) \to \mathcal{L}(\mathcal{H}_{F_c} \otimes \mathcal{H}_{F_t})]$, where the subscripts t and c denote a control and a target system, respectively, that maps any two quantum channels, $\widetilde{A} : \mathcal{L}(\mathcal{H}_{A_I}) \to \mathcal{L}(\mathcal{H}_{A_O})$ and $\widetilde{B} : \mathcal{L}(\mathcal{H}_{B_I}) \to \mathcal{L}(\mathcal{H}_{B_O})$, into another channel $\widetilde{C} : \mathcal{L}(\mathcal{H}_{P_t} \otimes \mathcal{H}_{P_c}) \to \mathcal{L}(\mathcal{H}_{F_t} \otimes \mathcal{H}_{F_c})$ defined below by its Kraus decomposition:

$$\widetilde{C}(\rho) = \widetilde{\widetilde{S}}_{\mathrm{QS}}(\widetilde{A} \otimes \widetilde{B})(\rho) \coloneqq \sum_{i,j} S_{ij}\, \rho\, S_{ij}^{\dagger}, \tag{42}$$

where

$$S_{ij} \coloneqq |0\rangle\!\langle 0| \otimes B_j A_i + |1\rangle\!\langle 1| \otimes A_i B_j, \tag{43}$$

where $\{A_i\}_i$ is a Kraus decomposition of quantum channel $\widetilde{A}$, that is, $\widetilde{A}(\rho_A) = \sum_i A_i\, \rho_A\, A_i^{\dagger}$, and analogously for $\{B_j\}_j$.

While in this form, the Kraus operators of the output channel of the quantum switch is defined as a function of the Kraus operators $\{S_{ij}\}_{ij}$ of the input channels of the quantum switch, they do not depend on which Kraus decomposition is given—that is, $\{S_{ij}\}_{ij}$ is the same for all possible $\{A_i\}_i$ and $\{B_j\}_j$. This is important to guarantee that the quantum switch is a well-defined higher-order map.

Note that if the input quantum channels are unitary channels $\widetilde{U}_A$ and $\widetilde{U}_B$, with single-operator Kraus decomposition U_A and U_B, then the output quantum channel of the quantum switch is also a unitary channel $\widetilde{U}_{\mathrm{QS}}$ given by

$$\widetilde{U}_{\mathrm{QS}}(\rho) = \widetilde{\widetilde{S}}_{\mathrm{QS}}(\widetilde{U}_A \otimes \widetilde{U}_B)(\rho) \coloneqq U_{\mathrm{QS}}\, \rho\, U_{\mathrm{QS}}^{\dagger}, \tag{44}$$

where

$$U_{\mathrm{QS}} = |0\rangle\!\langle 0| \otimes U_B U_A + |1\rangle\!\langle 1| \otimes U_A U_B, \tag{45}$$

where U_A and U_B and the unitary Kraus operators of $\widetilde{U}_A$ and $\widetilde{U}_B$, respectively.

The Choi operator S_{QS} of the quantum switch superchannel $\widetilde{\widetilde{S}}_{\mathrm{QS}}$ is defined as

$$S_{\mathrm{QS}} \coloneqq |w\rangle\!\langle w|\,, \tag{46}$$

where

$$|w\rangle \coloneqq |0\rangle_{P_c} |\mathbb{1}\rangle_{P_t A_I} |\mathbb{1}\rangle_{A_O B_I} |\mathbb{1}\rangle_{B_O F_t} |0\rangle_{F_c} + |1\rangle_{P_c} |\mathbb{1}\rangle_{P_t B_I} |\mathbb{1}\rangle_{B_O A_I} |\mathbb{1}\rangle_{A_O F_t} |1\rangle_{F_c} \tag{47}$$

where we recall that $|\mathbb{1}\rangle = \mathbb{1} \otimes \mathbb{1} \sum_i |ii\rangle = \sum_i |ii\rangle$ is the Choi vector of the identity channel and is proportional to a maximally entangled state.

It can be directly checked that S_{QS} satisfies Eqs. (33)–(38), which define a general two-slot superchannel. Hence, S_{QS} is a general two-slot superchannel that therefore maps all input quantum channels into a quantum channel even when acting on part of the input channels. However, by direct comparison with Eqs. (33)–(38), which define a sequential two-slot superchannel, it is possible to verify that the quantum switch is **not** a sequential two-slot superchannel. In fact, it can be checked that the quantum switch also cannot be decomposed into a convex combination of sequential superchannels with different fixed ordering of the slots, that is,

$$S_{\mathrm{QS}} \neq q\, S_{1 \prec 2} + (1-q)\, S_{2 \prec 1} \tag{48}$$

for all real numbers $q \in [0,1]$ and all sequential two-slot superchannels $S_{1\prec 2}$ and $S_{2\prec 1}$. Hence, **the**

quantum switch cannot be realized by a quantum circuit with open slots or by a probabilistic mixture of quantum circuits with open-slots that call the input channels in different orders.

The quantum switch has been interpreted as a higher-order transformation with **indefinite causal order**, in particular, with a quantum control of computational order: when the control quantum system is set to state $|+\rangle = 1/\sqrt{2}(|0\rangle + |1\rangle)$ (or any nontrivial superposition of $|0\rangle$ and $|1\rangle$), the state of the control becomes entangled with the order of the application of the input channels $\widetilde{A}$ and $\widetilde{B}$, which can hence be interpreted as a controlled coherent superposition of different orders.

3.1 The classical switch

The classical switch, also defined originally in Ref. [3], is the particular case of the quantum switch that acts on a classical (instead of a quantum) control system. This means that its Choi operator S_{CS} does not contain the terms of the Choi operator of the quantum switch that are off-diagonal in the computational basis of the control system, which are responsible for the coherence of the superposition. That is,

$$S_{\mathrm{CS}} \coloneqq |0\rangle\langle 0|_{P_c} \, |\mathbb{1}\rangle\langle\mathbb{1}|_{P_t A_I} \, |\mathbb{1}\rangle\langle\mathbb{1}|_{A_O B_I} \, |\mathbb{1}\rangle\langle\mathbb{1}|_{B_O F_t} \, |0\rangle\langle 0|_{F_c} + |1\rangle\langle 1|_{P_c} \, |\mathbb{1}\rangle\langle\mathbb{1}|_{P_t B_I} \, |\mathbb{1}\rangle\langle\mathbb{1}|_{B_O A_I} \, |\mathbb{1}\rangle\langle\mathbb{1}|_{A_O F_t} \, |1\rangle\langle 1|_{F_c} \tag{49}$$

Similarly to the quantum switch, it can also be checked that the classical switch (i) is a general two-slot superchannel and (ii) cannot be decomposed as a convex combination of sequential two-slot superchannels of different orders. Hence, **the classical switch also cannot be implemented by quantum circuits**.

At the same time, the classical switch is not interpreted as a higher-order transformation with indefinite causal order, but rather as an object that, although it does not fit into the standard quantum circuit model, describes a computation with classical control of orders that is causally well defined.

4 Many-slot superchannels

In general, many-slot superchannels can be categorized into different subsets, with different physically relevant properties. Below we briefly mention them, and point towards the references that discuss them in more detail, with precise definitions.

- Parallel [4–6]: Straightforward generalization of the parallel two-slot superchannels presented here, these correspond to all superchannels that can be implemented by quantum circuits with an open slot where all input quantum channels can be plugged in in parallel.

- Sequential [4–6]: Straightforward generalization of the sequential two-slot superchannels presented here, these correspond to all superchannels that can be implemented by quantum circuits with open slots. They are widely referred to as *quantum combs* and have also been called Quantum Circuits with Fixed Order (QCFO).

- ConvSequential: The convex hull of the sets of sequential superchannels in different fixed orders; represents all superchannels that can be expressed as a classical mixture of quantum circuits that apply the input quantum channels in different, fixed orders.

- QCCC [7, 8]: The acronym stands for Quantum Circuits with Classical Control. Represents classical control of computational order, generalizing the idea of the classical switch.

- QCQC [7, 8]: The acronym stands for Quantum Circuits with Quantum Control. Represents quantum control of computational order, generalizing the idea of the quantum switch. Smallest subset of superchannels presented here to contain elements that display an indefinite causal order. Arguably the largest set of superchannels that is physically well understood, although the possibility of their experimental implementation remains an open question.

- Purify [1]: Called purifiable or unitarily extensible superchannels, these are the many-slot superchannels whose Choi operators can be written as the partial trace of a rank-1 superchannel in a higher-dimensional space. Note that the Choi operator of *every* superchannel can be expressed as the partial trace of a rank-1 *operator* in a higher-dimensional space, however, crucially, it is *not always* the case that this rank-1 operator itself corresponds to the Choi operator of a superchannel (this is only always true in the case of the single-slot superchannel). They accept an interpretation of corresponding to reversible transformations in a larger Hilbert space, and as such have been argued to be potentially "physical".

- General [1–3, 9]: These are all higher-order transformations that satisfy Eq. (1). While mathematically well defined, the superchannels in this set that do not belong to any of the other subsets are not physically well understood. Some examples are somewhat understood operationally, in terms of the violation of causal inequalities, such as defined in Ref. [9], one of the foundational works of the field.

These subsets of superchannels obey the following strict hierarchy:

$$\mathsf{Parallel} \subset \mathsf{Sequential} \subset \mathsf{ConvSequential} \subset \mathsf{QCCC} \subset \mathsf{QCQC} \subset \mathsf{Purify} \subset \mathsf{General}. \tag{50}$$

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