Higher-order quantum transformations: From quantum circuit design to indefinite causal order

Tutorial I. Case study: Quantum channel discrimination as a higher-order problem

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Exercise I. Choi operators and link product

(1) Let the Choi operator of a linear map $\widetilde{C}:\mathcal{L}(\mathcal{H}_I)\to\mathcal{L}(\mathcal{H}_O)$ be defined as

$$C=\sum_{ij}|i\rangle\langle j|\otimes\widetilde{C}(|i\rangle\langle j|),$$

where $C\in\mathcal{L}(H_I\otimes\mathcal{H}_O)$. Calculate the Choi operator of the following maps:

a. Trace: $\Lambda_1(\rho)=\mathrm{tr}(\rho)$.

b. Transposition: $\Lambda_2(\rho)=\rho^T$.

c. Identity channel: $\Lambda_3(\rho)=\rho$.

d. Fully depolarizing channel: $\Lambda_4(\rho)=\mathrm{tr}(\rho)\,\frac{\mathbb{1}_O}{d_O}$.

e. Fully dephasing channel: $\Lambda_5(\rho)=\sum_a \mathrm{tr}(|a\rangle\langle a|\,\rho)\,|a\rangle\langle a|$ (computational basis).

f. Measure-and-reprepare channel channel: $\Lambda_5(\rho)=\sum_a \mathrm{tr}(M_a\rho)\sigma_a$

(2) Let $\mathcal{J}:[\mathcal{L}(\mathcal{H}_I)\to\mathcal{L}(\mathcal{H}_O)]\to\mathcal{L}(H_I\otimes\mathcal{H}_O)$ be the higher-order map that transforms linear maps into their Choi operators, i.e., $\mathcal{J}(\widetilde{C})=C$.

Given maps $\widetilde{F}:\mathcal{L}(\mathcal{H}_1)\to\mathcal{L}(\mathcal{H}_2)$ and $\widetilde{G}:\mathcal{L}(\mathcal{H}_2)\to\mathcal{L}(\mathcal{H}_3)$, with respective Choi operators $F\in\mathcal{L}(\mathcal{H}_1\otimes\mathcal{H}_2)$ and $G\in\mathcal{L}(\mathcal{H}_2\otimes\mathcal{H}_3)$, show that

$$\mathcal{J}(\widetilde{G}\circ\widetilde{F})=F*G.$$

(3) Show that

a. $\widetilde{C}$ is CP $\implies C\geq 0$.

b. $\widetilde{C}$ is TP $\implies \mathrm{tr}_O C=\mathbb{1}_I$.

c. If C is the Choi operator of $\widetilde{C}$, then $\mathrm{tr}_I[(\rho^T\otimes\mathbb{1}_O)C]=\widetilde{C}(\rho)$.

Exercise II. Quantum testers

(1) How would you "measure" a quantum channel using a quantum state ρ and a quantum measurement $\{M_i\}_i$? Write the quantum circuit that implements a measurement on a quantum channel $\widetilde{C}$ and the expression for the probability $p(i)$ of obtaining classical outcome i as a function of the map $\widetilde{C}$.

(2) Rewrite $p(i)$ as a function of the Choi operator C of $\widetilde{C}$, quantum state ρ, and quantum measurement $\{M_i\}$ using the link product.

(3) From a higher-order operations perspective, define the most general set of higher-order maps $\{\widetilde{\widetilde{T}}_i\}_i$, called a quantum tester, that transforms a quantum channel into a probability distribution. Rewrite $p(i)$ as a function of the Choi operator C of $\widetilde{C}$ and the Choi operators $\{T_i\}$ of $\{\widetilde{\widetilde{T}}_i\}_i$ using the link product. Compare it with the expression for $p(i)$ in part **(2)**.
Hint: This transformation will be described by a set of higher-order maps $\{\widetilde{\widetilde{T}}_i\}_i$, $\widetilde{\widetilde{T}}_i : [\mathcal{L}(\mathcal{H}_I) \to \mathcal{L}(\mathcal{H}_O)] \to \mathbb{C}$, with each $\widetilde{\widetilde{T}}_i$ mapping $\widetilde{C}$ to an element $p(i)$ of a probability distribution.

(4) Prove that a set of operators $\{T_i\}_i$, $T_i \in \mathcal{L}(\mathcal{H}_I \otimes \mathcal{H}_O)$ corresponds to the Choi operator of a quantum tester if and only if it satisfies

$$T_i \geq 0 \quad \forall i \tag{1}$$

$$\mathrm{tr}\left(\sum_i T_i\right) = d_O \tag{2}$$

$$\sum_i T_i = {}_O\sum_i T_i, \tag{3}$$

where ${}_OX := \mathrm{tr}_O(X) \otimes \frac{\mathbb{1}_O}{d_O}$ is the trace-and-replace map acting on $\mathcal{L}(\mathcal{H}_O)$.
Hint: Write the Choi operator of TP maps according to the parametrization $C = X - {}_OX + \frac{\mathbb{1}_{IO}}{d_O}$ for $X = X^\dagger$.

(5) Prove that every quantum tester can be implemented by a quantum state ρ and a quantum measurement $\{M_i\}_i$ in a quantum circuit. That is, that for every set of operators $\{T_i\}_i$ that satisfies the conditions in part **(4)**, there exists a quantum state ρ and a quantum measurement $\{M_i\}$ such that $T_i = \rho * M_i^T$.
Hint: Consider the Ansatz

$$\rho := \left(\mathbb{1}^I \otimes \sqrt{\sigma}^T\right) \sum_{ij} |ii\rangle\langle jj| \left(\mathbb{1}^I \otimes \sqrt{\sigma}^T\right)$$

and

$$M_i^T := \left(\sqrt{\sigma}^{-1} \otimes \mathbb{1}^O\right) T_i \left(\sqrt{\sigma}^{-1} \otimes \mathbb{1}^O\right),$$

where $\sigma := \mathrm{tr}_O \sum_i T_i / d_O$.

(6) Given an ensemble of quantum channels $\mathcal{E} = \{p_i, C_i\}_i$, write the optimization problem corresponding to finding the maximal average probability of successfully discriminating the ensemble

a. as a function of ρ and $\{M_i\}_i$;

b. as a function of $\{T_i\}_i$.